HIGH FREQUENCY TECHNIQUES
An Introduction to RF and Microwave Engineering

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CHAPTER 4

Distributed Circuits

4.1 TRANSMISSION LINES

A single inch of wire can have 10 nH of inductance, an impedance of \( +j63 \, \Omega \) at 1 GHz. Interconnecting circuit parts with uncontrolled wire lengths would lead to large, unwanted reactances. It would also lead to unpredictable circuit behavior because circuit components having transmission paths larger than about one tenth of a wavelength impart not only reactive changes but resistive transformations in impedance as well.

Today, with computer assistance, it is possible to analyze circuits having nearly arbitrary geometry, and therefore any hookup wire lengths theoretically can be handled analytically. Nevertheless, it is inadvisable to use irregular transmission line geometries except when there is a specific good reason for doing so.

In the early 1930s computers were not available to analyze circuits, and it was necessary that engineers confine transmission circuit designs to simple, regular cross sections in order that performance could be predicted. These were called uniform transmission lines, and they permitted more predictable behavior for distributed circuits.

Even today, for distributed circuits, wherein the lengths of the transmission paths are appreciable compared to the operating wavelength, it is generally advisable to interconnect circuits and circuit parts with uniform transmission lines. Unless a transmission line is kept uniform in cross section, its effect on the circuit becomes impractically difficult to compute by hand, and it is equally difficult to gain insight into how the circuit works.

A transmission line is a set of conductors that are long compared to a wavelength (generally considered to be \( >\lambda/10 \)) and have a uniform cross section along their length for which a characteristic impedance \( Z_0 \) can be defined.

In this chapter we first describe without proof the behavior of signals on transmission lines in order to provide a perspective of their operation. Later these transmission line circuit behaviors are derived analytically.
The lumped circuit model for a lossless, balanced, uniform transmission line is shown in Figure 4.1-1. In the equivalent circuit, $L$ is the inductance/unit length and $C$ is the capacitance/unit length of the uniform transmission line. For the lossless case the characteristic impedance $Z_0$ of the line is given by

$$Z_0 = \sqrt{\frac{L}{C}} \text{ (Ω)}$$

where $Z_0$ is defined as the ratio of voltage-to-current for a traveling wave in either direction. For the lossless line $Z_0$ is independent of frequency, a fact that is most important for broadband matching. When a transmission line is terminated in its characteristic impedance ($Z_0$), no reflections occur. That is, all power traveling toward the load is absorbed at the load, and none is reflected back toward the generator. There are other ways to accomplish total absorption of a traveling wave by the load, and we shall examine them as matching techniques.

The transmission line shown in Figure 4.1-2 is an example of a balanced line because both of its conductors have the same impedance to ground. For this reason stray voltage and currents induced onto either conductor of the line from interfering sources tend to be identical on both conductors, producing common mode voltage on the conductors and canceling insofar as the signal is concerned. The signal voltage is the difference in potential between the two conductors, and accordingly is called the differential mode voltage.

A balanced line example is the twisted pair phone line. The twist is added to further ensure that the two lines have identical impedances to ground, and, consequently, noise pickup is almost entirely in the common mode. This noise is not induced onto the signals carried by the conductors since the signals are impressed on the line in the differential mode.

Another example of a two-wire balanced transmission line is the twin lead used between an outside roof antenna and a TV set. Its geometry yields a characteristic impedance of about 300 Ω, more closely matching that of the source impedance of television antennas. Its balanced nature provides the same noise immunity to such sources as automobile ignitions, motors, and other
noise sources. However, it cannot provide immunity to noise that enters the system through radiation reception by the television aerial itself.

The balanced line pair can be further protected from interference by enclosing it within a shield, in which case the line is a balanced shielded transmission line.

For actual radio circuits unbalanced transmission lines, one of whose conductors is generally at ground potential, are more common (Fig. 4.1-3). The primary examples are stripline and microstrip, requiring only a “center conductor” pattern that can be photoetched for low cost and easy reproducibility.

Coaxial lines (also called coax) were developed to provide a flexible line with inherent shielding. They are used throughout the microwave bands, for example, up to 1 GHz for television cable systems and to 50 GHz or more in instrumentation applications.

Stripline evolved from coax in about 1950 and was considered a breakthrough due to its manufacturing ease. Its lack of mechanical flexibility was not a disadvantage since it was generally used to interconnect circuit parts installed on a common circuit board.

Microstrip evolved after stripline. It shared the reproducibility advantages of stripline but had the further advantage of providing easy access to circuit com-

Figure 4.1-2  Examples of balanced transmission lines.

Figure 4.1-3  Unbalanced transmission line formats commonly used in wireless applications.
ponents since there was no upper ground plane to get in the way of component installation and tuning. However, microstrip’s format usually results in an inhomogeneous dielectric medium, that is, a differing dielectric media for various portions of the transmission line’s cross section. This occurs because microstrip is usually implemented as a conductor pattern on a dielectric material with air dielectric above the transmission line patterns.

As frequency increases, the capacitance between center conductor and ground causes an increasingly larger percentage of the energy to propagate within the dielectric, resulting in a greater signal delay for higher frequencies than for lower frequencies. This variation in signal delay with frequency distorts wideband signals because their various frequency components experience differing transmission delays through the inhomogeneous transmission line medium.

4.2 WAVELENGTH IN A DIELECTRIC

The wavelength is the distance a sinusoid propagates in its period. Wavelength is reduced in proportion to the square root of the dielectric constant $\varepsilon_R$ of the propagating medium, since the speed of propagation is reduced in that proportion (Fig. 4.2-1). For a nonmagnetic dielectric

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_R}} \quad (4.2-1a)$$

$$v = \frac{c}{\sqrt{\varepsilon_R}} \quad (4.2-1b)$$

$$f\lambda = v \quad (4.2-1c)$$

$$f\lambda_0 = c \quad (4.2-1d)$$

Figure 4.2-1 Effect of dielectric material is to decrease propagation speed, and with it, wavelength.
where $\lambda_0$ is the free space wavelength, $c$ is the velocity of light in free space ($2.9979 \times 10^8$ m/s), and $\lambda$ is the wavelength in the dielectric medium.

Typical transmission line dielectrics range from pure Teflon with relative dielectric constant, $\varepsilon_R = 2.03$, to alumina ($\text{Al}_2\text{O}_3$) with $\varepsilon_R = 10$. Sapphire is the single-crystalline form of alumina and is an ideal, albeit expensive, microwave dielectric. Ceramic materials are available with relative dielectric constants up to 100 for use as primary resonators for oscillator circuits. Pure water has a dielectric constant of 81 [1] at RF and microwave frequencies.

Teflon material has very low dissipative loss and is a primary choice for microwave circuit boards. However, in pure form, it both expands and cold flows substantially with temperature and therefore is usually reinforced with embedded glass fibers for dimensional stability. The glass often is woven into a cloth fabric for reinforcing printed circuit (PC) boards. The presence of glass fibers increases the dielectric constant, and, if the fibers have a particular average orientation, as occurs with woven fabrics, the result is a dielectric constant that is a function of direction in the dielectric. Such a material is called an anisotropic dielectric.

To overcome the directional variation of dielectric constant the dielectric is sometimes reinforced using small, randomly oriented glass fibers (microfibers), making it uniform in all directions. A popular microfiber circuit board material is Rogers Duroid.

### 4.3 PULSES ON TRANSMISSION LINES

The concept of reflections on transmission lines can be visualized by considering the reaction of an open-circuited transmission line when a voltage pulse is sent down it.

Consider a voltage pulse applied to a lossless transmission line. The pulse propagates at the speed of light in a transmission line with air or vacuum surrounding the conductors or propagates at reduced speed if the line is embedded within a higher dielectric constant medium. For a lossless line, the pulse propagates along the line, as depicted in sketches 1 to 4 in Figure 4.3-1, its voltage undiminished with distance since there are assumed to be no dissipative or radiative losses. Associated with the voltage wave is a current wave. The ratio of voltage to current is $Z_0$.

In this theoretical idealization, on reaching the open-circuited end, all of the incident energy must be reflected since it can neither be radiated nor stored there. Of course, on a real open-circuited transmission line, some dissipative and radiative losses would occur, but we ignore them in this example.

Since all of the power of the pulse must be reflected at the open end of the line, the situation is equivalent to the introduction of a reverse-going pulse that is otherwise identical to the incident pulse. At the open circuit the current must be zero. To satisfy this condition, the reflected current must be directed opposite to the incident current. This requires that the associated reflected voltage
have the same polarity as the incident voltage, which results in a doubling of the voltage at the end of the line.

The total voltage on the line is equal to the sum of the incident and reverse-going pulses (sketches 4 to 7 of Fig. 4.3-1). Since the reflected voltage wave has the same amplitude as the incident wave, reflection from an open-circuit terminated line produces a doubling of the voltage at the end of the line as the voltages of the incident and reflected waves add together. The current at the end of the line is the sum of an incident current pulse and an equal magnitude but oppositely directed reflected current pulse, resulting in a net current of zero. We would expect this at an open-circuit termination.

After reflection at the open-circuit end of the line, the pulse travels back toward its source. At the source the pulse could again reflect if the source impedance is not equal to the line’s characteristic impedance \( Z_0 \). This pulse example is useful in demonstrating physically how the doubling of voltage occurs at the open-circuit termination of a transmission line. Similar reasoning shows that the current at a short-circuit termination of the transmission line produces a doubling of current at the line’s end and a zero voltage.

### 4.4 INCIDENT AND REFLECTED WAVES

Just as pulses on a transmission line that is open circuited produce a doubling of the incident voltage, so also do continuous sine waves when applied to the same transmission line termination condition.
In this case, however, the sine wave, of voltage $V_I$, is continuously applied at the source end. The reflected voltage $V_R$, an identical sine wave propagating back toward the generator, combines with it to produce a standing-wave pattern, creating voltage maxima when $V_I$ and $V_R$ are in phase and minima when they are out of phase (Fig. 4.4-1). The peak voltage magnitude of the standing-wave pattern is shown in Figure 4.4-1b. The maxima and minima are separated by a half wavelength due to the doubling of their relative velocities resulting from their opposite directions of travel.

In general, the net voltage at any point on the line is the result of the phasor addition of $V_I$ and $V_R$ at that point. Note that the amplitude of the standing-wave pattern is not a sine wave. It is the envelope of the peak AC voltage as a function of distance along the line. Such voltage (and the associated current) standing waves are analogous to those of waves in water that reflect from a seawall or other reflecting obstacle. In this example of a lossless line terminated in an open circuit, the peak voltage is $2V_A$ and the minimum voltage is zero. At the voltage nulls the current is $2I_A$ where $I_A = V_A/Z_0$. This will be seen from the analysis of transmission lines to follow.
4.5 REFLECTION COEFFICIENT

In the open-circuit load, lossless line case, the magnitudes of incident and reflected voltages are equal, and at the “load” end of the line they are in phase. This produces the voltage doubling there. But for arbitrary loads, $V_I$ and $V_R$ are neither equal in magnitude nor do they have the same phase. We define a complex reflection coefficient at any point, $x$, on the line as

$$\Gamma(x) = \frac{V_R(x)}{V_I(x)} = \rho \angle \phi$$

(4.5-1)

where $V_R$ and $V_I$ are reflected and incident phasor voltages, respectively.

On a lossless line the respective magnitudes of $V_I$ and $V_R$ do not change, thus the magnitude of the reflection coefficient, $\rho$, does not change either. Only the angle $\phi$ changes as the two waves travel by each other in opposite directions; $\phi$ depends upon the relative phases of $V_R$ and $V_I$ at the load as well as the electrical distance from the load.

For example, returning to the open-circuited case in Figure 4.4-1, since $\phi_{LOAD} = 0^\circ$, $V_{LOAD-INC} + V_{LOAD-REF} = 2$, a voltage peak occurs at the load. On the other hand, if there is a short circuit at the load, the total voltage at the end of the line must be zero. Then $\phi_{LOAD} = 180^\circ$, $V_{LOAD-INC} + V_{LOAD-REF} = 0$, and there is a voltage null at the load position.

In general, a line is terminated in some complex load, and the incident and reflected voltages have neither the same magnitudes nor are they precisely in or out of phase at the line’s terminus. The magnitudes of the voltage peak and null values are

$$V_{MAX} = V_{Incident} (1 + \rho)$$

(4.5-2a)

$$V_{MIN} = V_{Incident} (1 - \rho)$$

(4.5-2b)

where $\rho$ is the ratio of the magnitude of the reflected wave to that of the incident wave. One of the earliest microwave measurement techniques consists of cutting a narrow longitudinal slot in a transmission line and moving a rectifying diode along the line to determine $|V_{MAX}|$ and $|V_{MIN}|$ as well as the locations of these maximum and minimum voltages relative to the load position. This is called a slotted line measurement. Prior to the development of the network analyzer (about 1965) this was the only practical way to measure microwave impedance. Special microwave oscillators having a 1-kHz amplitude modulation were used in conjunction with standing-wave ratio (SWR) meters, consisting of tuned and calibrated 1-kHz amplifiers, to make the measurement more sensitive and precise. The ratio of the maximum and minimum voltage amplitudes was and continues to be called the voltage standing-wave ratio, or $VSWR$, sometimes just SWR:
\[ \text{VSWR} = \frac{|V_{\text{MAX}}|}{|V_{\text{MIN}}|} = \frac{1 + \rho}{1 - \rho} \quad (4.5-3a) \]

and conversely

\[ \rho = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \quad (4.5-3b) \]

### 4.6 RETURN LOSS

Relative to the incident power, return loss is the fraction of power returned from the load:

\[ \text{Return loss} = \rho^2 \text{ (fraction)} = 20 \log \rho \text{ (dB)} \quad (4.6-1) \]

Within the industry, VSWR and return loss are used as alternative means for specifying match, even though the slotted line is rarely if ever used for measurements. A perfect match of the load to the transmission line occurs when

\[ Z_L = Z_0 \quad (4.6-2) \]

This condition produces no reflected wave, hence \( \rho = 0 \) and \( \text{VSWR} = 1 \). Then all incident power is absorbed in the load; no power is reflected, and, when expressed in decibels, the return loss is \(-\infty\).

### 4.7 MISMATCH LOSS

Relative to the incident power, mismatch loss is the fraction of power absorbed (not returned) from the load:

\[ \text{Mismatch loss} = 1 - \rho^2 \text{ (fraction)} = 10 \log(1 - \rho^2) \text{ (dB)} \quad (4.7-1) \]

*In fractional form, return loss and mismatch loss sum to unity.* Care should be taken to study these definitions. It is a common error to confuse them. Table 4.7-1 lists values of return loss and mismatch loss for various values of the reflection coefficient, \( \rho \).

As an example, if the VSWR of a load is 1.5, \( \rho \) is 0.2, return loss is 0.04, and mismatch loss is 0.96. This means that the load absorbs 96% of the incident power and 4% is reflected. In decibels, the return loss is 14 dB and the mismatch loss is 0.18 dB. Notice that the minus sign for the losses in decibels is dropped conversationally because it is implied in the term “loss.” However, when using a network simulator, losses in decibels must be entered as negative quantities.
In Sections 4.6 and 4.7 return loss and mismatch loss were defined in terms of reflections at a mismatched load. However, these terms also can be applied to the input of a two-port network terminated by a matched or mismatched load. For example, a reactance connected in shunt with an otherwise match-terminated transmission line produces a reflection. For this circuit return loss and mismatch loss can be calculated.

Now, if there are two or more sources of reflection on the line (Fig. 4.8-1), the resulting reflections can combine to produce an overall reflection that depends not only upon the individual reflections but their reflective interactions as

![Diagram](image)

**Figure 4.8-1** Interaction between two reflecting networks separated by transmission line length.
well. To evaluate this combination, suppose two circuits, having $\Gamma_1$ and $\Gamma_2$
complex reflection coefficients are interconnected by a lossless transmission line
of electrical length $\theta$. For this discussion it is assumed that generator and load
are matched to each other and to the characteristic impedance of the line, in
this case normalized to $Z_0 = 1$.

Due to the multiple reflections between them and their spacing, the obstacles
will interact in a manner that causes their total insertion loss to vary. The loss
of the combination may be higher or lower than their simple loss, that is, the sum of their individual mismatch losses that would be measured when they are
separately connected between generator and load.

The amount by which the actual loss differs from the simple loss is called the
mismatch error, usually expressed in decibels. The term mismatch error arises
because, if we ignore their interaction and estimate their combined loss to be
their simple loss, we would encounter an error equal to this mismatch error
value. To analyze this reflection interaction it is useful to define the transmission coefficient $T$.

At the point of reflection on a transmission line, we defined an incident
voltage $V_I$ and a reflected voltage $V_R$. The phasor sum of these two voltages is
the total voltage $V_T$. This is the voltage on the line at the point of reflection:

$$V_T = V_I + V_R$$

The transmission coefficient $T$ is the ratio of $V_T$ to $V_I$, just as the reflection
coefficient $\Gamma$ is the ratio of $V_R$ to $V_I$. Then it follows that

$$T = 1 + \Gamma$$

where both $T$ and $\Gamma$ are complex numbers. Note that $T$ can be greater than
unity. In fact, this occurs at the end of an open-circuited transmission line at
which $\Gamma = 1$ and $T = 2$.

The situation when two sources of reflection exist on a transmission line is
shown in Figure 4.8-2. An incident voltage $V_I = 1$ encounters the first obstacle
and the voltage at the first obstacle is $T_1$. This voltage proceeds on toward the
second obstacle. When $T_1$ reaches the second obstacle, it is further affected. In

Figure 4.8-2  The reflection interaction between two circuits separated by lossless line
length [1, p 407].
the absence of reflections between the obstacles, the resulting voltage at the second obstacle would be $V_2$ (expected). This voltage launches a wave traveling toward the load as

$$V_2 \text{ (expected)} = T_1 T_2 e^{-j\theta} \quad (4.8-3)$$

The “expected” insertion loss (IL), or simple loss, of the two obstacles, neglecting their reflective interactions, would be the sum of their individual mismatch losses or

$$\text{IL \ (expected)} = 20 \log |T_1|^{-1} + 20 \log |T_2|^{-1} \quad (4.8-4)$$

where the negative exponents are used in order that IL is a positive number when expressed in decibels.

However, when $T_1$ reaches the second obstacle, a part of its energy is reflected by it. This reflected wave travels back toward the first obstacle at which a third reflection occurs. The process continues in a manner similar to that of the multiple reflections occurring between facing, partially reflecting mirrors, creating an infinite series of reflections and re-reflections.

The initial waves and the multiple reflections are as described in Figure 4.8-2. Each time the reflections make a round trip between obstacles, they gain an additional phase change of $e^{-2j\theta}$ as well as the factor $\Gamma_1 \Gamma_2$. Each such round trip path adds another term to the expression for the $V_2$ wave traveling toward the load. The result is an infinite series of terms:

$$\frac{V_2}{V_1} = T_1 T_2 e^{-j\theta} [1 + \Gamma_1 \Gamma_2 e^{-2j\theta} + (\Gamma_1 \Gamma_2 e^{-2j\theta})^2 + \cdots] \quad (4.8-5)$$

The magnitude of the quantity outside the brackets, $|T_1 T_2|$, is the simple loss ratio. The quantity inside the square brackets, $1 + \Gamma_1 \Gamma_2 e^{-2j\theta} + (\Gamma_1 \Gamma_2 e^{-2j\theta})^2 + \cdots$, is the effect of the reflective interaction between the two obstacles on the transmission line. Fortunately, this infinite series has a closed form [2, pp. 2-3], namely

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1 \quad (4.8-6)$$

Because $|\Gamma_1 \Gamma_2 e^{-2j\theta}| < 1$ for passive obstacles, (4.8-5) reduces to

$$\frac{V_2}{V_1} = T_1 T_2 e^{-j\theta} [1 + \Gamma_1 \Gamma_2 e^{-2j\theta}]^{-1} \quad (4.8-7)$$

Since insertion loss is $\text{IL} = 20 \log |V_1/V_2|$, the actual insertion loss for the circuit in Figure 4.8-2 is

$$\text{IL \ (actual)} = 20 \log |T_1|^{-1} + 20 \log |T_2|^{-1} + |\text{ME}| \quad (\text{dB}) \quad (4.8-8)$$
where ME is called the mismatch error and given by

\[
ME = 20 \log_{10}|1 - \Gamma_1 \Gamma_2 e^{-j2\phi}| \text{ (dB)} 
\]  

(4.8-9)

The mismatch error can either increase or decrease the base insertion loss of the two obstacles. The extreme values for ME are given, respectively, by using the + and – options. That is,

\[
ME \text{ (extreme values)} = 20 \log|1 \pm \rho_1 \rho_2| 
\]  

(4.8-10)

where \(\rho_1\) and \(\rho_2\) are the magnitudes of the reflection coefficients.

The larger the product of the reflection coefficients, the larger the mismatch error, and accordingly the larger the spread between maximum and minimum mismatch loss values. Customarily, it is usually the extreme mismatch error values that are called the mismatch errors because they are the error limits that can be experienced by ignoring the interaction of two reflections spaced along a transmission line. Notice that the mismatch error occurs between each pair of obstacles. If there are three obstacles spaced along a transmission line, there can be two separate mismatch errors.

A mismatch error can occur between a practical generator (or measurement system) and the reflections of a device under test if the generator is not perfectly matched to the transmission line. Mismatch errors also occur between connectors when neither has a unity VSWR and/or when one or more imperfectly matched connectors interact with an imperfectly matched device to which they are attached. In summary a mismatch error can arise whenever two causes of reflection are spaced along a transmission system.

Mismatch errors are especially large with cascaded reactive filters because these circuits accomplish their filtering action by reflecting power in the stopband. For example, suppose that two reactive filters are connected in cascade, each having 20 dB of isolation (when installed between matched source and load) at a frequency we wish to block. We would expect, neglecting reflective interaction, to obtain 40 dB of isolation in the stopband, the sum of the separate isolation values of the two filters. However, to provide 20 dB of isolation reactively, each filter must have \(\rho = 0.995\), or \(\rho^2 = 0.99\). For the pair of filters the mismatch error limits are

\[
ME_+ = 20 \log(1 + 0.99) = +6 \text{ dB}  
\]  

(4.8-11a)

\[
ME_- = 20 \log(1 - 0.99) = -40 \text{ dB}  
\]  

(4.8-11b)

In the first case, \(ME_+\) has the same sign as the base losses. Thus, under ideal spacing of the filters, we can obtain 46 dB of stopband isolation. This is 6 dB greater isolation than expected from the simple addition of the individual isolation values. However, in the second case, \(ME_-\), which occurs with the least optimal spacing, placing two lossless, reactive 20-dB filters in cascade may yield
0 dB of total attenuation. Nor is this unfavorable spacing unlikely since the stopband may represent a very wide frequency bandwidth over which the least favorable electrical spacing is likely to occur at various stopband frequencies.

Of course, real filters, even if designed as reactive networks, have finite loss, and the observation of 0 dB loss for a pair of them is a limiting, but unrealizable condition. Nevertheless, the calculation shown above suggests that a substantial reduction in isolation from that expected when ignoring VSWR interaction is readily possible, dependent upon their electrical spacing at each stopband frequency.

Interestingly, there is another way of treating an interacting pair of mismatches using their VSWR values [3, Example 4.5, 8]. The overall VSWR due to a pair of dissipationless mismatches having individual VSWR values $\text{SWR}_1$ and $\text{SWR}_2$ (where $\text{SWR}_1 > \text{SWR}_2$) that are referenced to and separated by a length of lossless $Z_0$ line can range from $\text{SWR}_1/\text{SWR}_2$ to $(\text{SWR}_1/\text{SWR}_2)$.

Applying this to the previous example, for a reactive filter providing 20 dB of isolation, $1/(1 - \rho^2) = 100$, therefore $\rho = 0.994987$ and the corresponding $\text{SWR}_1 = \text{SWR}_2 = 397.997487$. Since the VSWRs are equal, $\text{SWR}_1/\text{SWR}_2 = 1$ and their minimum insertion loss is 0 dB. The maximum VSWR = $(397.99)^2 = 158,402$. The corresponding $\rho = 0.999987$. The return loss is $1/(1 - \rho^2) = 39,601 = 45.98$ dB, which probably agrees with the prior estimates of 0 and 46 dB within the accuracy of the significant figures carried in the calculations.

4.9 THE TELEGRAPHER EQUATIONS

To establish a mathematical basis for the analysis of transmission lines, we shall use a uniform line equivalent circuit that includes line losses, as shown in Figure 4.9-1. Note that, while the series elements are shown only in the upper conductor, this equivalent circuit is applicable to both balanced and unbalanced transmission lines. Also, this analysis applies to any waveforms that propagate on the transmission line, not just sinusoidal excitations.

In this equivalent circuit $R$, $L$, $G$, and $C$ are, respectively, the series resistance, series inductance, shunt conductance, and shunt capacitance per unit length.
length of the transmission line. The small section of transmission line can be analyzed by applying Kirchhoff’s laws and taking the limit as this circuit length tends to zero to obtain the differential equations for the circuit [4, p. 23]. Kirchhoff’s voltage law requires that the sum of the voltages about a closed loop is zero. Thus

\[ v(x, t) - R \Delta x i(x, t) - L \Delta x \frac{\partial i(x, t)}{\partial t} - v(x + \Delta x, t) = 0 \]  \hspace{1cm} (4.9-1)

Similarly, applying Kirchhoff’s current law, which requires that the sum of the currents into a circuit node be zero, gives

\[ i(x, t) - G \Delta x v(x + \Delta x, t) - C \Delta x \frac{\partial v(x + \Delta x, t)}{\partial t} - i(x + \Delta x, t) = 0 \]  \hspace{1cm} (4.9-2)

For the above two equations, divide all terms by \( \Delta x \) and let \( \Delta x \to 0 \). Note that as \( \Delta x \to 0 \)

\[ v(x + \Delta x, t) \to v(x, t) \]
\[ i(x + \Delta x, t) \to i(x, t) \]

Also,

\[ \frac{v(x + \Delta x, t) - v(x, t)}{\Delta x} \to \frac{\partial v(x, t)}{\partial t} \]
\[ \frac{i(x + \Delta x, t) - i(x, t)}{\Delta x} \to \frac{\partial i(x, t)}{\partial t} \]

The resulting differential equations in (4.9-3) are the time-domain form of the transmission line equations and are called the telegrapher equations, presumably because their initial use was in the analysis and design of long telegraph lines:

\[ \frac{\partial v(x, t)}{\partial x} = -Ri(x, t) - L \frac{\partial i(x, t)}{\partial t} \]  \hspace{1cm} (4.9-3a)
\[ \frac{\partial i(x, t)}{\partial x} = -Gv(x, t) - C \frac{\partial v(x, t)}{\partial t} \]  \hspace{1cm} (4.9-3b)

### 4.10 TRANSMISSION LINE WAVE EQUATIONS

These telegrapher equations apply for any time-varying \( v(x, t) \) and \( i(x, t) \) on the transmission line. In the steady-state sinusoidally excited case, they reduce to
\[
\frac{dV(x)}{dx} = -(R + j\omega L)I(x) \quad (4.10-1a)
\]
\[
\frac{dI(x)}{dx} = -(G + j\omega C)V(x) \quad (4.10-1b)
\]

in which \(R, L, G,\) and \(C\) are per unit length and \(V(x)\) and \(I(x)\) are the phasor forms of \(v(x, t)\) and \(i(x, t)\), whose time variation is implicit. These two equations can be solved simultaneously to yield two equations, one a function of only \(V\) and the other a function of only \(I\). To do so, for example, differentiate the second equation with respect to \(x\) to get

\[
\frac{d^2I(x)}{dx^2} = -(G + j\omega C)\frac{dV(x)}{dx}
\]

or

\[
\frac{dV(x)}{dx} = -\frac{1}{G + j\omega C} \frac{d^2I(x)}{dx^2} \quad (4.10-2)
\]

Then substitute this value for \(dV(x)/dx\) into (4.10-1a).

\[
\frac{d^2I(x)}{dx^2} - (R + j\omega L)(G + j\omega C)I(x) = 0
\]

or

\[
\frac{d^2I(x)}{dx^2} - \gamma^2 I(x) = 0 \quad (4.10-3a)
\]

Using the same procedure, (4.10-1b) yields

\[
\frac{d^2V(x)}{dx^2} - \gamma^2 V(x) = 0 \quad (4.10-3b)
\]

in which we define

\[
\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (4.10-4)
\]

Solving differential equations involves removing the differentials (integrating) by guessing or recognizing the solution from prior differentiation experience. The only functions that are unchanged after double differentiation are the exponentials, \(e^{\pm \gamma x}\). (A close possibility would have been \(\sin \gamma x,\) however, twice differentiating \(\sin \gamma x\) would give \(-\sin \gamma x\).) A second-order differential equation such as this should have two independent solutions. In this case the
two solutions correspond to the two values for the ± sign, they are

\[
\frac{d^2 e^{\gamma x}}{dx^2} = \gamma^2 e^{\gamma x} \quad \text{and} \quad \frac{d^2 e^{-\gamma x}}{dx^2} = \gamma^2 e^{-\gamma x}
\]

Accordingly, the sum of these two solutions represents the complete solution to either (4.10-3a) or (4.10-3b).

\[
V(x) = V_I e^{-\gamma x} + V_R e^{\gamma x} \quad (4.10-5a)
\]

\[
I(x) = I_I e^{-\gamma x} - I_R e^{\gamma x} \quad (4.10-5b)
\]

where \( \gamma \) is the propagation constant for the voltage and current waves. Use of the negative sign before \( I_R \) in (4.10-5b) results in the same value for the reflection coefficient for \( I \) as for \( V \). Physically the two terms in each equation correspond to waves propagating, respectively, in the +x and −x directions. These two equations are called the \textit{wave equations} for a uniform transmission line.

### 4.11 Wave Propagation

The terms with negative exponents (such as \( V_I e^{-\gamma x} \)) correspond to a wave propagating in the +x direction, from left to right in Figure 4.9-1. Since the generator is attached at the left and the load at the right in Figure 4.9-1, this corresponds to an \textit{incident wave}, hence the subscript \( I \). Similarly, the positive exponent terms (such as \( V_R e^{\gamma x} \)) correspond to a wave traveling in the −x direction and represent a \textit{reflected wave} in our convention. These use the subscript \( R \).

The total voltage at any \( x \) location on the line is the phasor sum of the incident and reflected voltages, \( V(x) = V_I(x) + V_R(x) \) and \( I(x) = I_I(x) - I_R(x) \). We can cast the relationship between \( V(x) \) and \( I(x) \) into the format of impedance by resorting to their interrelation expressed in either (4.10-1a) or (4.10-1b). For example, invoking (4.10-1a) and substituting into it the value for \( V(x) \) given by (4.10-5a) gives

\[
I(x) = \frac{1}{R + j\omega L} \frac{dV(x)}{dx} = -\frac{1}{R + j\omega L} \frac{d(V_I e^{-\gamma x} + V_R e^{\gamma x})}{dx}
\]

\[
I(x) = \frac{\gamma}{R + j\omega L} (V_I e^{-\gamma x} - V_R e^{\gamma x}) \quad (4.11-1)
\]

From this we see that the current, \( I(x) \) has terms that are of the same form as those of the voltage \( V(x) \), but there is a negative sign between them. This is because the current convention for reflected waves reverses while that for voltages does not.
The factor \( \frac{R + j\omega L}{\gamma} \) has the dimensions of impedance and is called the \textit{characteristic impedance} \( Z_0 \) of the transmission line. It can be rewritten as

\[
Z_0 = \frac{R + j\omega L}{\gamma} = \frac{R + j\omega L}{\sqrt{(R + j\omega L)(G + j\omega C)}} \tag{4.11-2}
\]

With this definition, the equation for \( I(x) \) has exactly the form of Ohm’s law, noting that the minus sign between terms accounts for the reversal of current direction for a reflected wave. In other words, \( Z_0 \) is the \textit{ratio of the respective voltage to current of waves traveling in either direction}:

\[
I(x) = I_I(x) - I_R(x) = \frac{V_I}{Z_0} e^{-\gamma x} - \frac{V_R}{Z_0} e^{+\gamma x} \tag{4.11-3}
\]

Notice that for a lossy line, for which \( R \) and \( G \) are nonzero, the characteristic impedance is a complex quantity that varies with frequency. However, for the \textit{lossless line}, the characteristic impedance is a constant, independent of frequency and given by

\[
Z_0 = \sqrt{\frac{L}{C}} \tag{4.11-4}
\]

For high frequency, low-loss transmission lines, (4.11-4) is a good approximation, since \( \omega L \gg R \) and \( \omega C \gg G \). For a transmission line with loss, the propagation constant \( \gamma \) is a complex quantity. Conventionally this is written with real and imaginary parts as

\[
\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{4.11-5}
\]

where \( \alpha \) and \( \beta \) are, respectively, the attenuation and phase constants of the transmission line with units of reciprocal length. The real term, \( \alpha \), is responsible for loss and is expressed in neper/\textit{unit length}; and the imaginary quantity, \( \beta \), is responsible for phase change and is expressed in radian/\textit{unit length}.

For a transmission line having loss, both the incident and reflected waves diminish in amplitude as they travel along the line due to dissipative and/or radiative losses. For an incident wave the amplitude diminishes with \( x \) according to the factor \( e^{-\alpha x} \). For example, if \( \alpha \) were 0.1 neper/meter, then a wave traveling 1 m would have its amplitude decreased by the factor \( e^{-0.1} = 0.9 \). That is, the voltage of the traveling wave would be reduced by 10%. Since power is proportional to the square of the voltage, the power loss would be about 19% or 0.8686 dB for this 1-m length of line.
Notice that this example describes the conversion between loss in nepers and decibels, namely \( \text{Loss (dB) = 8.686 loss (nepers)} \) conversely \( \text{Loss (nepers) = 0.115 loss (dB)} \). These conversions are listed in the inside cover summaries for convenient reference.

As noted, the imaginary portion of the propagation constant \( \beta \) produces a phase change in the wave as a function of distance. It has the dimensions of \( \text{radians/unit length} \) or, optionally, \( \text{degrees/unit length} \). An any instant of time, with time frozen so to speak, a snapshot of the voltage amplitude along the line of the incident wave (assuming there is no reflected wave for this example) would be a sinusoid. The distance between adjacent wave crests, or between adjacent troughs, or between any other similar pair of corresponding points of the wave sinusoid is the wavelength \( \lambda \). Since the sinusoid completes a cycle in \( 2\pi \) radians, it follows that

\[
\beta = \frac{2\pi}{\lambda} \text{ radians/unit length} \tag{4.11-6a}
\]

\[
\beta = \frac{360^\circ}{\lambda} \text{ degrees/unit length} \tag{4.11-6b}
\]

For the lossless line,

\[
\alpha = 0 \tag{4.11-7a}
\]

\[
\beta = \omega\sqrt{LC} \tag{4.11-7b}
\]

Usually one does not know explicitly the distributed \( L \) and \( C \) values for a transmission line. Conventionally, the transmission line \( Z_0 \) is provided, along with the effective relative dielectric constant \( e_R \) of the line. Therefore, it is more convenient to determine electrical length by first determining the wavelength on the line, finding the fraction of a wavelength of the circuit in question, and multiplying either by \( 2\pi \) for length in radians or by \( 360^\circ \) to obtain electrical length in degrees. This is demonstrated by the following example.

Electrical length can be expressed in either radians or degrees. For example, consider a 6 in. length \( (x = 6 \text{ in.}) \) of coaxial cable having a polyethylene insulator between its inner and outer conductors, for which the relative dielectric constant is 2.26. What is the cable’s electrical length \( \theta \) at 1 GHz? Since the wavelength in free space for a 1-GHz sinusoid is 11.8 in.,

\[
\theta = \beta x = \frac{x}{\lambda} 360^\circ = \frac{x}{\lambda_0/\sqrt{e_R}} 360^\circ = \frac{6 \text{ in.}}{11.8 \text{ in./} \sqrt{2.26}} 360^\circ = 275^\circ \tag{4.11-8}
\]

Since \( 2\pi \) radians corresponds to \( 360^\circ \), the electrical length in radians is

\[
\theta = 275^\circ \cdot \frac{6.28}{360^\circ} = 4.80 \text{ rad} \tag{4.11-9}
\]
Notice that it was not necessary to determine $\beta$ explicitly. However, since the wavelength in the cable at 1 GHz is 7.85 in., then

$$\beta = \frac{360^\circ}{7.85 \text{ in.}} = 45.9^\circ/\text{in. at 1 GHz} \tag{4.11-10}$$

Note from (4.11-7b) that $\beta$ is frequency dependent. In a nondispersive transmission media (for which electrical length is proportional to frequency), $\beta$ is directly proportional to frequency. Thus, at 1.2 GHz the same cable would have a propagation constant of

$$\beta = \frac{1.2 \text{ GHz}}{1 \text{ GHz}} \times \frac{45.9^\circ/\text{in.}}{55.1^\circ/\text{in. at 1.2 GHz}} = \frac{1.2 \text{ GHz}}{1 \text{ GHz}} \times \frac{45.9^\circ/\text{in.}}{55.1^\circ/\text{in. at 1.2 GHz}} \tag{4.11-11}$$

and the same 6-in. cable would have an electrical length of 330° or 5.76 rad at 1.2 GHz.

### 4.12 PHASE AND GROUP VELOCITIES

The incident wave, traveling from the generator toward the load, is, from (4.10-5a)

$$V(x)_{\text{Incident}} = V_I e^{-\gamma x} \tag{4.12-1}$$

in which $V_I$ is a phasor with implicit sinusoidal time variation. Written explicitly

$$v(x)_I = \text{Re}[V_I e^{-\gamma x} e^{j\omega t}] = Ce^{-\alpha x} \cos(\omega t - \beta x + \phi) \tag{4.12-2}$$

where $C$ is a constant amplitude factor and $\phi$ is the initial phase of the wave at $t = x = 0$. This is interpreted as a sinusoidal wave propagating in the $+x$ direction with an exponential attenuation due to the factor $\alpha$. The phase constant $\beta$ is given in (4.11-6), namely, $\beta = 2\pi/\lambda$. A similar equation can be written for the reflected wave that travels back toward the generator in the $-x$ direction. The voltage $v(x, t)_I$ has a constant phase for

$$\beta x = \omega t \quad \text{or} \quad x = \frac{\omega t}{\beta} \tag{4.12-3}$$

Therefore, a constant phase point on the traveling wave moves at a phase velocity given by $dx/dt$ or

$$v_p = \frac{\omega}{\beta} \tag{4.12-4}$$
Thus, the voltage on the transmission line for either an incident or reflected wave varies as [4, pp. 45–46]

\[ e^{j(\omega t - \beta x)} = e^{j\omega t / v_P} \]  (4.12-5)

Any periodic function of time can be represented as a sum of sinusoidal waves using a Fourier analysis (see Section 7.27). If \( v_P \) is the same for all of them, then their addition will faithfully reproduce along the line the temporal wave shape at the input, only delayed by the time of propagation \( x/v_P \). This is the case for a lossless transmission line having \( g = j b = \sqrt{L/C} \) for which

\[ v_P = \frac{1}{\sqrt{L/C}} \]  (4.12-6)

For lossless transverse electromagnetic (TEM) mode transmission lines, the velocity at which a constant phase, \( v_P \), moves on the line and the velocity at which a packet of information, such as a modulation envelope, moves, \( v_G \), are identical and simply equal to the velocity of propagation, \( v \).

This produces a convenient means for determining the characteristic impedance of the TEM mode for a lossless line. Since

\[ Z_0 = \sqrt{\frac{L}{C}} \]

and

\[ v = \frac{1}{\sqrt{L/C}} \]

Therefore,

\[ Z_0 = \frac{1}{v C} \]  (4.12-7)

This equation provides a convenient expression for determining the characteristic impedance of a TEM mode, lossless transmission line if its distributed capacitance and velocity of propagation are known.

However, for transmission lines having losses and for waveguides \( v_P \) does vary with frequency, dispersion occurs, and the signal waveform at the input will be distorted as its separate frequency components travel along the line at different velocities. If the dispersion is significant, it may not be meaningful to speak of a single propagation delay for a signal waveform. However, if the dispersion is moderate over the signal frequency bandwidth, an approximate group velocity \( v_G \) can be defined [4]. To determine its value, suppose that an
incident voltage wave, \( v_1 \), consists of two frequencies that are traveling on the line, one lower and one higher in frequency than \( \omega_0 \), respectively, having a combined value at \( x = 0 \) given by

\[
v_I(x = 0, t) = A[\sin(\omega_0 - d\omega) t + \sin(\omega_0 + d\omega) t]
\]  
(4.12-8)

Then, at a general point \( x \) on the line

\[
v_I(x, t) = A\{\sin[(\omega_0 - d\omega) t - (\beta_0 - d\beta) x] + \sin[(\omega_0 + d\omega) t - (\beta_0 + d\beta) x]\}
\]

in which \( \beta \) is assumed to be a function of \( \omega \) and hence is the reason for the differential \( d\beta \). Rearranging terms in the arguments gives

\[
v_I(x, t) = A\{\sin[(\omega_0 - d\omega) t - (\beta_0 - d\beta) x] + \sin[(\omega_0 t - \beta_0 x) + (d\omega t - d\beta x)]\}
\]  
(4.12-9)

Next, recognizing that the two sine terms are of the form \( \sin(A - B) \) and \( \sin(A + B) \), expanding them and simplifying gives

\[
v_I(x, t) = 2A \cos(d\omega t - d\beta x) \sin(\omega_0 t - \beta_0 x)
\]  
(4.12-10)

This reveals that the total voltage on the line corresponds to a high frequency sinusoid varying at \( \omega t \) rate whose amplitude is modulated by

\[
\cos[(d\omega) t - (d\beta)x]
\]

which varies with both time and distance. The modulation, itself, has the properties of a traveling wave. A constant phase of the modulation occurs for

\[
(d\omega)t = (d\beta)x
\]  
(4.12-11)

from which a group velocity, \( v_G = x/t \), can be defined as

\[
v_G = \frac{d\omega}{d\beta}
\]  
(4.12-12)

This can be related to the phase velocity \( v_P = \omega/\beta \) as follows:

\[
v_G = \frac{d\omega}{d\beta} = \frac{d\omega}{d(\omega/v_P)} = \frac{1}{(d/d\omega)(\omega/v_P)} = \frac{1}{[v_P - \omega(dv_P/d\omega)]/v_P^2}
\]

\[
v_G = \frac{v_P}{1 - (\omega/v_P)(dv_P/d\omega)}
\]  
(4.12-13)

This result was derived for only two frequencies, but using similar reasoning it can be argued that a signal comprising many individual frequencies will travel with an envelope velocity approximately equal to \( v_G \), if the dispersion is not
too great. However, this will not be true for lines having large dispersion, but in those cases it is inappropriate to assign any velocity to such a modulation because its shape would be changing so rapidly as it propagates. The concept of group velocity also is not meaningful when \( \frac{dv_P}{d\omega} \) is positive, a situation called anomalous dispersion, \[4, \text{p. 48}\] because then, as can be seen from (4.12-13) group velocity would appear to be infinite or negative.

### 4.13 REFLECTION COEFFICIENT AND IMPEDANCE

We can evaluate the reflection coefficient in terms of the load impedance by dividing (4.10-5a) by (4.11-3). For this evaluation let us choose \( x = 0 \) at the load, then at the load location the exponential functions equal one and the load impedance \( Z_L \) is given by

\[
Z_L = Z_L \frac{1 + V_R/V_I}{1 + V_R/V_I} = Z_L \frac{1 + V_R/V_I}{1 - V_R/V_I} \tag{4.13-1}
\]

where \( V_I \) and \( V_R \) are phasor quantities. But \( \Gamma(x) = V_R(x)/V_I(x) \) is the definition of the reflection coefficient and therefore

\[
Z_L = Z_0 \frac{1 + G_L}{1 - G_L} \tag{4.13-2}
\]

where, in general, \( \Gamma \) and \( G_L \) are complex numbers. Equivalently,

\[
\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Y_0 - Y_L}{Y_0 + Y_L} \tag{4.13-3}
\]

where \( Y_0 = 1/Z_0 \). From this we see that there will be no reflected wave if \( Z_L = Z_0 \), that is, when the load impedance is equal to the characteristic impedance of the line. Under this condition, all of the power incident on the line will be absorbed in the load.

Since the choice of \( x = 0 \) is arbitrary, (4.13-2) must apply for any \( x \). It follows that

\[
Z(x) = Z_0 \frac{1 + \Gamma(x)}{1 - \Gamma(x)} \tag{4.13-4}
\]

and

\[
\Gamma(x) = \frac{Z(x) - Z_0}{Z(x) + Z_0} = \frac{Y_0 - Y(x)}{Y_0 + Y(x)} \tag{4.13-5}
\]

for all \( x \).
4.14 IMPEDANCE TRANSFORMATION EQUATION

One of the most common tasks in microwave engineering is the determination of how a load impedance \(Z_L\) is transformed to a new input impedance \(Z_{IN}\) by a length of uniform transmission line of characteristic impedance \(Z_0\) and electrical length \(\theta\) (Fig. 4.14-1).

To simplify this derivation, we assume that the line length is lossless. With the choice of \(x = 0\) at the load, the input to the line is at \(x = -l\), and the input impedance there is

\[
Z_{IN} = Z(x = -l) = \frac{V(x = -l)}{I(x = -l)} = \frac{e^{j\beta l} + \Gamma_L e^{-j\beta l}}{e^{j\beta l} - \Gamma_L e^{-j\beta l}} \quad (4.14-1)
\]

Now substitute \(\Gamma_L = (Z_L - Z_0)/(Z_L + Z_0)\), \(\beta l = \theta\), the identities \(e^{j\beta l} = \cos \beta l + j \sin \beta l\) and \(e^{-j\beta l} = \cos \beta l - j \sin \beta l\) into (4.14-1), and remove canceling terms to get

\[
Z_{IN} = Z_0 \left[ \frac{2Z_L \cos \theta + jZ_0 \sin \theta}{2Z_0 \cos \theta + j2Z_L \sin \theta} \right] \quad (4.14-2)

\]

Similar reasoning can be used to evaluate the input impedance when the transmission line has finite losses. The result is

\[
Z_{IN} = Z_0 \left[ \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right] \quad (4.14-3)
\]

This is one of the most important equations in microwave engineering and is called the impedance transformation equation. Remarkably, (4.14-2) and (4.14-3) have exactly the same format when derived in terms of admittance. For the lossless line

\[
Y_{IN} = Y_0 \left[ \frac{Y_L + jY_0 \tan \theta}{Y_0 + jY_L \tan \theta} \right] \quad (4.14-4)
\]

Figure 4.14-1 Equivalent circuit of lossless uniform transmission line of electrical length \(\theta\) and characteristic impedance \(Z_0\) terminated in impedance \(Z_L\).
and for the line with loss

\[
Y_{IN} = Y_0 \left[ \frac{Y_L + Y_0 \tanh \gamma l}{Y_0 + Y_L \tanh \gamma l} \right]
\]  

(4.14-5)

where \(Y_0 = 1/Z_0\), \(Y_L = 1/Z_L\), and \(Y_{IN} = 1/Z_{IN}\). Expressions (4.14-4) and (4.14-5) can be verified by substituting these equivalences, respectively, into (4.14-2) and (4.14-3).

In general, even the lossless expressions of (4.14-2) and (4.14-4) are complex to apply, as will be seen by a subsequent example. Beside the fact that when \(Z_L = Z_0\) the line is always matched and \(Z_{IN} = Z_0\), there are four additional cases that are both easy to apply and worthy of note.

**Case 1** Lossless lines of zero length or multiples of a half wavelength present the load impedance unchanged at their input terminals:

\[
Z_{IN} = Z_L
\]  

(4.14-6)

If the transmission line’s length is zero or an integer number of half wavelengths \((0^\circ, 180^\circ, 360^\circ, 540^\circ, \ldots)\), then \(\tan \theta = 0\), and (4.14-2) reduces to \(Z_{IN} = Z_L\). That is, whatever impedance terminates, the line will be presented unchanged at the input.

**Case 2** A short-circuit-terminated lossless transmission line is inductive for \(\theta\) up to \(90^\circ\), becomes an open circuit when \(\theta = 90^\circ\), and is capacitive for \(90^\circ < \theta < 180^\circ\). Thereafter the behavior is repeated every half wavelength. For the short-circuit termination (4.14-2) reduces to

\[
Z_{IN} = jZ_0 \tan \theta
\]  

(4.14-7a)

A plot of the input reactance \((Z_0 \tan \theta)\) for \(Z_0 = 1\) is shown in Figure 4.14-2. The line length is chosen to be \(90^\circ\) at 1000 MHz. Notice that when the electrical length of the line is \(45^\circ\) (at 500 MHz) the reactance is inductive and numerically equal to \(Z_0\), while at \(135^\circ\) (1500 MHz) it is capacitive and the reactance is numerically equal to \(-Z_0\).

The fact that a shorted quarter wavelength line \((\theta = 90^\circ)\) becomes an open circuit (at 1000 MHz in Fig. 4.14-2) is useful in rotary joints for antennas, bias injection and filter networks, and a variety of other applications.

Notice also that if the shorted line section is short compared to a wavelength, \(\tan \theta \approx \theta\) (in radians) and (4.14-2) can be approximated as

\[
Z_{IN} \approx jZ_0 \theta = j\omega \left(\frac{Z_0 l}{v}\right)
\]  

(4.14-7b)

since \(\theta = \beta l = (2\pi/\lambda)l = 2\pi(f/v)l = \omega l/v\). For most transmission lines \(\theta\) is
linearly proportional to frequency, as is the impedance of an inductor, $Z = j\omega L$. Therefore a short length of short-circuited terminated transmission line behaves as an inductor of value

$$L_{\text{EFF}} \approx \frac{\theta Z_0}{\omega v} \quad (0 \leq 0.53 \text{ rad}) \quad (4.14-7c)$$

For example, if a 1-cm length of $Z_0 = 120 \ \Omega$, air coax line is terminated in a short:

$$L_{\text{EFF}} \approx \frac{Z_0 l}{v} = \frac{120(1 \text{ cm})}{3 \times 10^{10} \ \text{cm/s}} = 4 \ \text{nH}$$

Since 1 cm $\approx 0.4$ in., the inductance of a high impedance air-dielectric line is approximately equal to 10 nH/in., the approximation presented earlier in (2.12-6).

The approximation (4.14-7c) also applies to a short length of $Z_0$ transmission line terminated in a load $Z_L$ provided that its characteristic impedance $Z_0 \gg |Z_L|$, as can be verified by applying this condition and $\tan \theta \approx \theta$ in (4.14-2).

The approximation $\tan \theta \approx \theta$ occurs frequently and has an error magnitude of less than 10% for values of $\theta$ up to 0.524 rad ($30^\circ$). A radian is $360^\circ/2\pi \approx 57.3^\circ$. Thus 0.5 rad is nearly $\frac{1}{12}$ of a wavelength. The values of $\theta$ and

**Figure 4.14-2** Reactance of lossless short-circuited transmission line versus frequency ($\theta = 90^\circ$ at 1 GHz).
tan θ are compared in Table 4.14-1 along with the size of the error in using this approximation, showing that the error magnitude is less than 10% at θ = 30° and diminishes rapidly for lesser values of θ.

**Case 3** An open-circuit terminated lossless transmission line is capacitive for θ up to 90°, becomes a short circuit when θ = 90°, and is inductive for 90° < θ < 180°. Thereafter the behavior is repeated every half wavelength. Thus, from (4.14-4) with $Y_L = 0$

\[ Y_{IN} = jY_0 \tan \theta \quad \text{or} \quad Z_{IN} = -jZ_0 \cot \theta \quad (4.14-8a) \]

This is plotted in Figure 4.14-3. Notice that the input reactance magnitude is capacitive and equal to $-Z_0$ when the line is 45° long (500 MHz). For θ = 90° (1000 MHz in Fig. 4.14-3) the open-circuit load is transformed to a short circuit at the input. It is inductive and equal to $Z_0$ when 135° long (1500 MHz). The input repeats every half wavelength after 180°. The facility for transforming an open circuit to a short circuit is equally useful in a variety of microwave circuits.

**TABLE 4.14-1** Approximating tan θ by Its Argument 0 (in radians)

<table>
<thead>
<tr>
<th>θ (deg)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ (rad)</td>
<td>0</td>
<td>0.175</td>
<td>0.349</td>
<td>0.436</td>
<td>0.524</td>
</tr>
<tr>
<td>tan θ</td>
<td>0</td>
<td>0.176</td>
<td>0.364</td>
<td>0.466</td>
<td>0.577</td>
</tr>
<tr>
<td>Error in tan θ ≈ θ</td>
<td>0</td>
<td>-0.6%</td>
<td>-4.1%</td>
<td>-6.4%</td>
<td>-9.1%</td>
</tr>
</tbody>
</table>

Figure 4.14-3 Normalized reactance of lossless open-circuited transmission line versus frequency (electrical length = 90° at 1 GHz).
By using similar reasoning to that applied to a short section of a shorted transmission line, and again applying the approximation \( \tan \theta \approx \theta \) for small values of \( \theta \), a short section of open-circuited transmission line has a capacitive admittance given by

\[
Y_{IN} \approx j Y_0 \theta \quad (4.14-8b)
\]

Since \( B_C = \omega C \) and \( \theta = \omega l/v \), the equivalent capacitance is

\[
C_{EFF} \approx \frac{\theta}{\omega Z_0} = \frac{l}{Z_0 v} \quad (\theta \leq 0.53 \text{ rad}) \quad (4.14-8c)
\]

and a reactance of

\[
\frac{1}{\omega C} \approx \frac{Z_0}{\theta} \quad (4.14-8d)
\]

For example, a section of 25-\( \Omega \) line that is 20\(^\circ\) long at 1 GHz has an equivalent capacitance of

\[
C_{EFF} \approx \frac{0.35 \text{ rad}}{6.28 \times 1 \times 10^9 \text{ Hz} \times 25 \Omega} = 2.2 \text{ pF}
\]

Its reactance \( X_C \) at 1 GHz is

\[
\frac{1}{\omega C_{EFF}} \approx \frac{25 \Omega}{0.35} = 71 \Omega
\]

If a short length of \( Y_0 \) line is connected to a load of admittance \( Y_L \), it behaves as a shunt capacitance approximated by (4.14-8c) provided that \( |Y_L| \ll Y_0 \) (i.e., \( |Z_L| \gg Z_0 \) and \( \tan \theta \approx \theta \)).

**Case 4** The normalized input impedance to a quarter wavelength long transmission line is the reciprocal of the normalized load impedance.

When the transmission line is a quarter wavelength long, \( \theta = 90^\circ \), and the input impedance to the line is

\[
Z_{IN} = Z_0 \frac{Z_0}{Z_L} \quad (4.14-9)
\]

Normalizing to \( Z_0 \) (dividing all impedances by \( Z_0 \)),

\[
z_{IN} = \frac{1}{Z_L} \quad (4.14-10)
\]
For this reason a quarter wavelength line is often called an impedance inverter. Notice that the normalized input impedance is equal to the normalized load admittance, that is,

\[ z_{\text{IN}} = \frac{1}{z_L} = y_L \quad (4.14-11) \]

where the normalized admittance \( y_L = Y_L / Y_0 \) and \( Y_0 = 1/Z_0 \).

Later we will see that this property allows us to convert between admittance and impedance using the Smith chart. Notice also that this impedance inverting property occurs not only for quarter wavelength length lines but for line lengths that are any odd multiple of a quarter wavelength \( (\lambda/4, 3\lambda/4, 5\lambda/4, \ldots) \).

In general, to transform any resistive load \( R_L \) to a desired input resistance \( R_{\text{IN}} \), one need only select a quarter wave line section having a \( Z_0 \) that satisfies

\[ Z_0 = \sqrt{R_{\text{IN}}R_L} \quad (4.14-12) \]

As an example of the resistive transformation of a 90° line length, consider a 50-Ω, quarter wavelength, transmission line terminated in 25 Ω. From (4.14-9) the input resistance is

\[ Z_{\text{IN}} = Z_0 \frac{Z_0}{Z_L} = 50 \frac{50}{25} = 100 \Omega \quad (4.14-13) \]

Thus, the 25-Ω resistive load has been “transformed” to an input resistance of 100 Ω at 1 GHz, and the variation of the input impedance with frequency for this case is shown in Figure 4.14-4.

It is a common but ill-advised practice to refer to the quarter wavelength transmission line as a quarter wave transformer because the inversion property can be used to transform a resistive load of one value to an input resistance of another value at a single frequency. However, this practice should be avoided. It is better to visualize it as a quarter wave impedance inverter, recognizing that even that property is limited to a single frequency of operation.

The inversion changes the real parts of the load, but in addition the sign of the imaginary part is also reversed, turning an impedance with an inductive part into one with a capacitive part or vice versa. For example, if in the prior case, rather than a purely resistive load of 25 Ω, we had instead a load of \( (25 + j25) \Ω \), the input impedance becomes

\[ Z_{\text{IN}} = Z_0 \frac{Z_0}{Z_L} = 50 \frac{50}{25 + j25} = 50 \frac{25}{25 + j25} = 25 - j25 \]

\[ = \frac{(2500)(25 - j25)}{625 + 625} = (50 - j50) \Omega \quad (4.14-14) \]

Notice that the transformation ratio of the real part is now 2 instead of the 4 obtained with a purely resistive load. Equally noteworthy, notice that the imaginary part of the input impedance is capacitive whereas the load’s imagi-
nary part was inductive! Thus the action of the quarter wavelength transmission line is quite different from that of an ideal impedance transformer, which simply multiplies any impedance by a constant factor.

Also, an ideal transformer, which is well approximated at low frequencies by a pair of magnetically coupled wire windings, has a fixed impedance transformation ratio that is frequency independent. On the other hand, the impedance transforming property of a quarter wave line section only approximates a constant transformation ratio for resistive loads and only over a narrow bandwidth centered at the design frequency as shown in Figure 4.14-4. Despite these limitations, the quarter wavelength line section has considerable practical utility for its transforming properties.

The preceding four special transmission line cases demonstrate the insight to be gained by application of the input impedance formula of (4.14-2). This formula is key to transmission line analyses. However, for complex load values and arbitrary line lengths, the formula can be quite tedious to apply. As an example, suppose that we wish to find the input impedance to a 50-Ω line that is 50° long and terminated in a load impedance of \((25 + j30)\) Ω (Fig. 4.14-5).
Applying (4.14-2) gives

\[
Z_{IN} = \frac{Z_L + jZ_0 \tan \theta}{Z_0 + jZ_L \tan \theta} = \frac{50 \cdot 25 + j30 + j50 \tan 50^\circ}{50 + j(25 + j30) \tan 50^\circ} = \frac{50 \cdot 25 + j89.6}{-14.2 + j29.8}
\]

\[
= 50 \frac{\sqrt{625 + 8027\angle74.4^\circ}}{\sqrt{202 + 888\angle64.5^\circ}} = 50 \frac{93.0\angle74.4^\circ}{33.0\angle64.5^\circ} = 141\angle10^\circ = (139 + j24) \Omega
\]

Notice that the 25-Ω real part of the load has been transformed to 139 Ω and that the +j30-Ω imaginary part of the load has been transformed to j24-Ω. From this it can be seen that, in general, the impedance transformation properties of a simple transmission line can be quite complex.

It is tedious to perform this calculation and very easy to make a mathematical error while doing so, thereby obtaining an erroneous result. Also, the calculation must be re-performed for each frequency of evaluation.

It would be a considerable challenge to predict, using only (4.14-2), how \(Z_{IN}\) varies with \(\theta\) as it is varied over, say, 0° to 180°. This evaluation would require charting \(Z_{IN}\) for numerous \(\theta\) values using the laborious calculation method described in (4.14-15). It was for this reason that Phillip Smith found an elegant graphical solution to determine input impedance, the subject of the next chapter.

### 4.15 IMPEDANCE MATCHING WITH ONE TRANSMISSION LINE

In the previous section it was shown that a length of transmission line can transform a real load resistance to one of a different value. Accordingly, one might ask whether any complex load impedance \(Z_L = R_L + jX_L\) can be con-
jugately matched to any other complex source impedance, \( Z_G = R_G + jX_G \), merely by proper selection of a line length of the appropriate \( Z_0 \) and electrical length \( \theta \). The following derivation (refer to Figure 4.15-1) yields the conditions under which this can be achieved.

From (4.14-2) with \( Z_{IN} \) set equal to the complex conjugate of \( Z_G \),

\[
Z_{IN} = R_G - jX_G = \frac{Z_L + jZ_0 \tan \theta}{Z_0 + jZ_L \tan \theta}
\]  (4.15-1)

Next, set the real and imaginary parts of this equation separately equal to each other and solve for \( Z_0 \) and \( \theta \). Note that \(-jX_G\) was used in the equation in order that \( Z_{IN} \) will be the complex conjugate of \( Z_G \):

\[
Z_0 = \sqrt{\frac{R_L(R_G^2 + X_G^2) - R_G(R_L^2 + X_L^2)}{R_G - R_L}}
\]  (4.15-2)

\[
\theta = \tan^{-1} \left[ \frac{Z_0(R_L - R_G)}{R_LX_G - R_GX_L} \right]
\]  (4.15-3)

The above equations are always mathematically valid, however, a practical solution exists only when the solution for \( Z_0 \) is a real and finite number.

### 4.16 FANO’S (AND BODE’S) LIMIT

#### Type A Mismatched Loads

Over a given bandwidth there is a definite limit to how well one can tune a mismatched termination. The most well known are called Fano’s limits. Fano derived these interesting and useful results within a thesis project [5].
One of Fano’s two integral equation limits applies to what he described as type A mismatched loads (Fig. 4.16-1). It is

\[
\int_0^\infty \ln \left| \frac{1}{\Gamma} \right| d\omega \leq \frac{\pi \omega_0}{Q} \quad (4.16-1)
\]

where the reflection coefficient \( \Gamma \) is referenced to \( Z_G \) and the real part of \( Z_G \) is \( R \), the same as the load resistance \( R \) in Figure 4.16-1.

Basically, (4.16-1) states that no matter what matching circuit is used, the area under the \( \ln(1/\Gamma) \) curve cannot exceed the value \( \pi \omega_0/Q \). Given this limit, we would usually prefer that \( \Gamma_{\text{MIN}} \) would be obtained over our specified band-width, \( \omega_1 - \omega_2 \), and then that \( \Gamma \) be unity for all other frequencies. This ideal matching requirement is shown in Figure 4.16-2.

In practice, of course, the matching behavior described in Figure 4.16-2 can only be approximated using real tuning circuits, but the ideal limit serves as an insightful restriction on how well a circuit can be tuned over a given band-width.

If we assume that the reflection coefficient is a constant equal to its minimum value in the passband, then the integral for the type A mismatched loads reduces to a constant multiplied by the frequency interval:

\[
\int_0^\infty \ln \left| \frac{1}{\Gamma} \right| d\omega = (\omega_2 - \omega_1) \ln \left| \frac{1}{\Gamma} \right| \quad (4.16-2)
\]

where \( \omega_1 \) is the frequency (in radians/second) at the lower band edge and \( \omega_2 \) is...
the upper band edge. We also define

\[
\text{Center frequency } = \omega_0 = \sqrt{\omega_1 \omega_2}
\]  
(4.16-3)

and

\[
\text{Fractional bandwidth } = \Delta \omega = \frac{\omega_2 - \omega_1}{\omega_0}
\]  
(4.16-4)

Then for the type A mismatched load, assuming that a minimum and constant value of reflection coefficient, \( \Gamma_{\text{MIN}} \), could be obtained throughout the passband \( \omega_1 - \omega_2 \)

\[
(\omega_2 - \omega_1) \ln \left| \frac{1}{\Gamma_{\text{MIN}}} \right| \leq \frac{\pi \omega_0}{Q_L}
\]  
(4.16-5)

Equivalently,

\[
|\Gamma_{\text{MIN}}| \geq e^{-\pi \omega_0/Q(\omega_2 - \omega_1)} = e^{-(\pi/Q)(f_0/\Delta f)}
\]  
(4.16-6)

where \( f_0 = \omega_0 / 2\pi \) and \( \Delta f = (\omega_2 - \omega_1) / 2\pi \).

For example, assume there is a type A circuit consisting of a 50-\( \Omega \) load resistor in parallel with a 10-pF capacitor to be matched to a 50-\( \Omega \) generator over the 700- to 1100-MHz band. Notice that 10 pF presents a large susceptance in parallel with the 50-\( \Omega \) load at the geometric center of this frequency band, 877.5 MHz. In reactance terms, at 877.5 MHz this is a capacitive reactance of only 18.1 \( \Omega \)!
The minimum reflection coefficient magnitude, \(|\Gamma_{\text{MIN}}|\), that we can expect to average over the band is found as follows.

First, calculate \(Q, f_0,\) and \(\Delta f\):

\[
f_0 = \sqrt{(700)(1100)} \, \text{MHz} = 877.5 \, \text{MHz}
\]

\[
X_C = \frac{159}{(10 \, \text{pF})(0.8775 \, \text{GHz})} = 18.1 \, \Omega
\]

\[
Q = \frac{R}{X_C} = \frac{50 \, \Omega}{18.1 \, \Omega} = 2.76
\]

\[
\Delta f = 700 - 1100 = 400 \, \text{MHz}
\]

Then

\[
|\Gamma_{\text{MIN}}| \geq e^{(-\pi/Q)(f_0/\Delta f)} = e^{(-3.14/2.76)(877.5/400)} = e^{-2.5} = 0.082 \tag{4.16-7}
\]

and the corresponding minimum average VSWR is

\[
\text{VSWR}_{\text{MIN}} = \frac{1 + |\Gamma_{\text{MIN}}|}{1 - |\Gamma_{\text{MIN}}|} = \frac{1 + 0.082}{1 - 0.082} = 1.18 \tag{4.16-8}
\]

To examine this case, a 10-element ladder network was used as a matching network [10, Sec. 5.3]. The element values were selected arbitrarily, 10 nH for inductors and 10 pF for capacitors, and the circuit optimized using a network simulator. The simulation goal was set equal to a 1.2 maximum VSWR over the 700 to 1100 MHz bandwidth. The result, obtained once the optimization appeared to produce little further improvement, is shown in Figure 4.16-3. An average VSWR of about 1.3 was obtained over the 700 to 1100 MHz bandwidth, comparing moderately well with the theoretical limit of 1.18.

**Type B Mismatched Loads**

Similar reasoning can be used with Fano’s type B mismatched loads (Fig. 4.16-4), for which Fano’s integral limit is

\[
\int_0^\infty \frac{1}{\omega^2} \ln \left| \frac{1}{\Gamma} \right| \, d\omega \leq \frac{Q}{\pi \omega_0} \tag{4.16-9}
\]

The limits for both type A and type B mismatched loads are commonly attributed to Fano [5]; however, Fano, himself, credits the limit for the type A circuits to Bode [7]. Whatever their origin, the limits give a useful insight into the problem of broadband matching.
Impedance Transformation Not Included

When Fano and Bode performed their research, presumably for the telephone industry, the frequencies of their concern were relatively low, a few megahertz. For this reason they probably considered that a simple change in resistive level could always be accommodated using an “ideal” transformer, which has no bandwidth limitations. Probably for this reason, they only considered the reactive part of the load as a mismatch with inherent bandwidth limitations. At
microwave frequencies the construction of a transformer that approximates frequency-independent behavior is not as practical. Thus, the usefulness of Fano’s limits at microwave frequencies are mainly confined to those cases for which the load has the correct resistive value \( R = Z_G \) but has some series or shunt reactance that must be tuned using a matching network.

**REFERENCES**


sistor amplifier design and evaluation, based on the S parameters of the transistor and the circuit that surrounds it.


EXERCISES

E4.1-1 A manufacturer of PIN switching diodes proposes to make a broadband switch module consisting of a PIN diode in shunt with a 50-Ω transmission line. When the diode is reverse biased, it appears as a 0.5-pF capacitor in shunt with the transmission line and is to pass signals with minimum reflection. He suggests to you that if the hookup wires (from the top of the diode to the center conductor of the line and shown as L1 and L2 above) are designed to have inductance values to satisfy

\[ Z_0 = \sqrt{\frac{L}{C}} = 50 \, \Omega \]

the module will be electrically indistinguishable from a length of 50-Ω line and consequently will provide matched transmission at all frequencies.

a. What should the hookup wire inductances \( L/2 \) each be made to satisfy the above relation?

b. Is it correct that this will provide an infinitely broadband transmission match?
c. Test your answer on a network simulator over the bandwidth 0 to 10 GHz.

E4.1-2 A 50-Ω air dielectric coaxial transmission line has an inner diameter of the outer conductor equal to 0.500 in. and a center conductor diameter of 0.217 in. What is its distributed capacitance per inch?

E4.4-1 A voltage wave with 100-V peak amplitude is incident on a transmission line extending in the z direction. When it reaches the load, one quarter of its power is reflected back toward the generator which is matched to the line.

a. What is the peak voltage on the line between generator and load?
b. What is the minimum voltage on the line between generator and load?
c. Sketch the manner in which the incident and reflected waves add and subtract along the line as z changes for a fixed time t.

E4.5-1

a. What is the magnitude of the reflection coefficient in E4.4-1?
b. What is the VSWR?

E4.5-2 A friend has just purchased a boat containing a marine radio. The radio has a built-in VSWR meter that indicates that the antenna has a VSWR of 3 to 1.

a. How much does this reduce the radiated power of the transmitter relative to that which would be radiated with a perfectly matched antenna?
b. What would you advise to tune the radio to the antenna?

E4.8-1 Show that for a susceptance \( jB \) connected in shunt with an otherwise matched transmission line of characteristic impedance \( Z_0 \), the magnitude of the transmission coefficient \( T \) is less than unity for all values of \( B \).

E4.8-2 You wish to send the output of a 500-MHz signal source to another building via a 50-Ω coaxial cable. The cable loss is 10 dB, which must be compensated by adding an amplifier at the output of the source. When connected to the 50-Ω cable, the source sees the cable as a reflection with VSWR = 10 (equivalently, the source output VSWR is 10), and you have a 20-dB amplifier with input VSWR of 20 to 1 referenced to 50 Ω. Neglecting other system interactions:

a. What will be the mismatch error in connecting this amplifier to the source?
b. Will it overcome the 10-dB cable loss in all cases?

E4.11-1 Show that the real part of the propagation constant, \( \alpha \), can be approximated for a low-loss transmission line having \( R \) ohms/unit length and no shunt conductance by
\[ a \approx \frac{R}{2Z_0} \]

**E4.11-2** Use the result you derived in Exercise 4.11-1 to estimate the loss in decibels at 1 GHz for 100 ft of JAN Type RG 224/U coaxial cable having a copper inner conductor of 0.106 in. diameter \((d_1)\) and a copper outer conductor with 0.370 in. diameter \((d_2)\). Assume the resistivity for copper is \(1.72 \times 10^{-8} \Omega \cdot \text{m}\) (then conductivity, \(\sigma = 5.8 \times 10^7 \text{ S/m}\)) and neglect the conduction losses through the dielectric. *Hint: Assume an equivalent current flow within one skin depth on the conductor surfaces.*

**E4.14-1** Beginning with (14.4-2) verify that

\[ Y_{\text{IN}} = Y_0 \left[ \frac{Y_L + jY_0 \tan \theta}{Y_0 + jY_L \tan \theta} \right] \]

**E4.14-2** Show that a *short length* of \(Z_0\) transmission line terminated by a load \(Z_L\) can be approximated as a series inductance \(L_{\text{EFF}} \approx Z_0 \theta / \omega\) when \(Z_0 \gg |Z_L|\).

**E4.14-3** A ham radio friend has purchased a 52-MHz transmitter with matched impedance antenna, both 50 \(\Omega\). He wishes to install the antenna on his roof and the transmitter in his study, a separation distance of about 25 ft. Unfortunately, he only has a 75-\(\Omega\) cable to make the hookup. The relative dielectric constant of the cable’s dielectric is 2.8. Is there any way that he can retain the very good match between transmitter and antenna using this cable?

**E4.14-4** A resourceful engineer found a way to connect a 12.5-\(\Omega\) load to a 50-\(\Omega\) generator without reflection when provided only with a spool of 50-\(\Omega\) cable. How did he do it?

**E4.14-5** An error was made in the design of an integrated circuit narrow-band amplifier with the result that its input impedance is \((50 - j50) \Omega\) instead of 50 \(\Omega\). Your company mounts the integrated chip on a “motherboard” printed circuit inside a housing with 50-\(\Omega\) connectors. Vibration specifications preclude installing a lumped coil at the input to tune the amplifier. You propose to print a section of 150-\(\Omega\) transmission line on the motherboard in cascade between the chip and input connector, reasoning that it will appear inductive and help to tune the mismatch. Specifications require an input VSWR of no more than 1.2.

**a.** What is the VSWR before this tuning?

**b.** How long should you make the 150-\(\Omega\) line (in degrees or wavelengths)?
c. What is the VSWR after the proposed tuning? Does it meet the objective of a VSWR = 1.2 maximum?

**E4.14-6** An engineer has a coil of coaxial cable and needs to determine its dielectric material for a thermal analysis. He has been told that the dielectric material is either Teflon \( (\varepsilon_R = 2.03) \) or polyethelene \( (\varepsilon_R = 2.26) \). He forms a line section \( D = 1 \) m long, and he places a short circuit at its load end. Then he connects the open end to a slotted line and examines the input impedance using a variable frequency generator. He finds that it has its first resonance (a short circuit at the input) at 105.28 MHz. Can you say what the dielectric material is?

**E4.15-1** Can you find a 50-\( \Omega \) perfectly matched input solution to E4.14-6 using a properly chosen \( Z_0 \) and electrical length? If so, what are the values?

**E4.15-2** A load impedance of \( 10 + j10 \) \( \Omega \) is to be matched to a 50-\( \Omega \) generator. Can this be done using only a section of transmission line in cascade with the load? What is its \( Z_0 \) and electrical length \( \theta \)?

**E4.16-1** An available power amplifier design is being considered for an application requiring that the input VSWR not exceed 1.5 over the 1700 to 2100 MHz band. The amplifier, however, has an input resistance of 50 \( \Omega \) that is shunted by 10 pF capacitance and presently does not meet this specification.

a. What is \( f_0 \) for the 1700 to 2100 MHz bandwidth?

b. What is the \( Q \) of this capacitively loaded input circuit?

c. What is Fano’s limit for the least VSWR obtainable over the 1700 to 2100 MHz bandwidth?

d. What value of input shunting inductance will parallel resonate the 10 pF. Use a network simulator to determine over what bandwidth the 1.5 VSWR can be achieved with this single shunt \( L \) element. Does this meet the amplifier requirement?

e. Using as many as six tuning elements and the network simulator optimizer, find a matching network that provides as low a VSWR over the 1700 to 2100 MHz bandwidth as you can achieve. Does it meet the maximum 1.5 VSWR over the 1.7 to 2.1 GHz bandwidth?
5.1 BASIS OF THE SMITH CHART

Knowledge of the basis and use of the Smith chart, a graphical presentation of the reflection coefficient with normalized impedance as a parameter overlay, is a sine qua non for the microwave engineer. Originally created as an aid to determining the input impedance to a transmission line, the Smith chart has become a universal aid to the design of matching circuits and to the display of measured data.

In the 1930s, when Phillip Smith [1–3] invented the graphical solution that facilitated the determination of how impedances were transformed by lengths of transmission lines, digital computers were unavailable to engineers. Engineers employed graphing strategies to gain insight into the variations produced by the independent variables of complex formulas. However, such insight was not forthcoming by attempting directly to graph the input impedance formula (4.14-2), repeated below, for a transmission line.

\[
Z_{IN} = Z_0 \left[ \frac{Z_L + jZ_0 \tan \theta}{Z_0 + jZ_L \tan \theta} \right]
\] (5.1-1)

The key to obtaining insight into this transformation was to recognize that, although impedance varies in a complex manner as one moves away from the load along a lossless transmission line, reflection coefficient variation is quite simple! In traveling from the load to the generator along a lossless line, only the angle of the reflection coefficient changes, not its magnitude (Fig. 5.1-1).

The reflection coefficient is the complex ratio of the reflected to the incident voltage waves. The total voltage on the line consists of the incident, \(V_I\), and the reflected, \(V_R\), voltages:

\[
V(x) = V_Ie^{-jbx} + V_Re^{+jbx}
\] (5.1-2)

\[
V(x) = V_Ie^{-j\theta} + V_Re^{+j\theta}
\] (5.1-2)

\[
\Gamma(x) = \frac{V_R(x)}{V_I(x)} = \rho L \Phi
\] (5.1-3)

---

*High Frequency Techniques: An Introduction to RF and Microwave Engineering*, By Joseph F. White.  
where \( V_R(x) = V_{Re}e^{xj\theta} \), \( V_I(x) = V_Ie^{-xj\theta} \), and \( V_R(0)/V_I(0) = \Gamma_{Load} \). Choosing \( x = 0 \) at the load and moving an electrical distance \( \theta \) toward the generator from the load causes the reflected wave \( V_R \) to have a negative change of phase, equal to \( \theta \), relative to its phase at the load. The incident wave \( V_I \) has a positive change of phase equal to \( \theta \), but the incident voltage is in the denominator, hence

\[
\Gamma = \Gamma_L e^{-2j\theta}
\]

(5.1-4)

Moving an electrical distance \( \theta \) from the load, the reflection coefficient argument is reduced by \(-2\theta\).

This is a key relationship. We will make use of it frequently as we use reflection coefficient change to determine the input impedance of a transmission line.

If the relationships of (4.13-4) and (4.13-5) are normalized to \( Z_0 \) (all impedances divided by \( Z_0 \)), \( \Gamma(x) \) and \( z(x) = Z(x)/Z_0 \) are related by

\[
z(x) = \frac{1 + \Gamma(x)}{1 - \Gamma(x)}
\]

(5.1-5)

\[
\Gamma(x) = \frac{z(x) - 1}{z(x) + 1}
\]

(5.1-6)

Both of these equations are bilinear transformations [4, p. 202]. That is, they are of the form

\[
w = \frac{Az + B}{Cz + D}
\]

(5.1-7)
where \( w \) and \( z \) are complex variables and \( A, B, C, \) and \( D \) are complex constants satisfying

\[
\Delta = AD - BC \neq 0
\]  

\[ (5.1-8) \]

The bilinear transformation is a one-to-one mapping that has been well studied in the theory of complex functions. In the case of (5.1-5) and (5.1-6) this means that: Every point in the \( z(\omega) \) plane corresponds to a unique point in the \( \Gamma(\omega) \) plane, and vice versa.

To illustrate this \( z \) and \( \Gamma \) correspondence, consider the example of the load impedance \( (25 + j30) \ \Omega \) terminating the 50° long, 50-\( \Omega \) line described in Figure 4.14-5 and evaluated in (4.14-13). Normalized, this impedance and the corresponding reflection coefficient at the load are

\[
z_L = 0.5 + j0.6
\]

\[ (5.1-9) \]

\[
\Gamma_L = \frac{z_L - 1}{z_L + 1} = \frac{0.5 + j0.6 - 1}{0.5 + j0.6 + 1} = 0.48 \angle 108^\circ
\]

\[ (5.1-10) \]

Let us now determine the impedance at a point 50 electrical degrees from the load. To do so we decrease the angle of the reflection coefficient by twice the electrical length (50°) of the transmission line, or 100°, to obtain the reflection coefficient at an electrical distance \( \theta = 50^\circ \) from the load:

\[
\Gamma_0 = \Gamma_L e^{-j\theta} = 0.48 \angle (108^\circ - 100^\circ) = 0.48 \angle 8^\circ
\]

\[ (5.1-11) \]

The corresponding normalized impedance is

\[
z_0 = \frac{1 + \Gamma_0}{1 - \Gamma_0} = \frac{1 + 0.475 + j0.07}{1 - 0.475 - j0.07} = 2.76 + j0.48
\]

\[ (5.1-12) \]

\[
Z_0 = z_0(50 \ \Omega) = (138 + j24) \ \Omega
\]

\[ (5.1-13) \]

These results are plotted in the \( \Gamma \) plane diagram in Figure 5.1-2.

Our strategy has succeeded. The load impedance \( 25 + j30 \ \Omega \) was normalized to the 50-\( \Omega \) transmission line and the corresponding reflection coefficient determined \( (\Gamma_L = 0.48 \angle 108^\circ) \). This reflection coefficient was transformed along the 50° long line by preserving its magnitude (0.48) and reducing its angle (108°) by twice the electrical length of the line (100°). The resulting reflection coefficient \( (\Gamma_L = 0.48 \angle 8^\circ) \) was used to calculate the normalized impedance \( (z_0 = 2.76 + j0.48) \) at the input to the line, and this value, when unnormalized to 50 \( \Omega \), gives the input impedance \( Z_0 = (138 + j24) \ \Omega \). This is close to the value of \( (139 + j24) \ \Omega \) obtained for the same example in (4.14-13) by using the impedance transformation formula. The small difference in the real part is likely due to round-off error. Notice that we are able to treat the electrical dis-
tance of the line to the load, \( \theta \), as a positive quantity, since the clockwise rotation of \( \Gamma \) takes its sign into account.

While successful, our strategy of using the simple rotation of \( \Gamma \) to calculate the impedance transformation of a load by a line length has not resulted in any real economy of computational effort due to the need to perform the complex transformations from \( z \) to \( \Gamma \) (5.1-6) and then from \( \Gamma \) to \( z \) (5.1-5).

However, we note that once a \( z \) plane point is mapped into the \( \Gamma \) plane it is done once and for all, since the \( z \) and \( \Gamma \) points have a one-to-one correspondence (Fig. 5.1-3). Thus, plotting normalized impedance values in the \( \Gamma \) plane in advance would permit entering the \( \Gamma \) unit circle directly with a normalized load impedance value and, after performing the required \( 2\theta \) clockwise rotation, reading the corresponding normalized input impedance to the line. This is the basis of the Smith chart.

However, performing this translation of the \( z \) to the \( \Gamma \) plane poses some obvious problems. First, the flags used to identify the corresponding \( z \) and \( \Gamma \) points take up so much room on the \( \Gamma \) chart that it would be impractical to show a sufficient number of them to permit accurate plotting of \( z \). Second, the \( z \) plane is semi-infinite for passive loads, corresponding to all impedances for which \( r \geq 0 \); hence an unduly large sheet of paper would be required to ac-

**Figure 5.1-2** The \( \Gamma \) plane representation of load impedance \( z_L \) normalized to 50 \( \Omega \) and transformed through a 50° length of 50-\( \Omega \) transmission line.
commodate large impedances in the $z$ plane. We note, however, that no such problem exists in the $\Gamma$ plane, wherein all passive loads must yield a $|\Gamma| \leq 1$, thereby fitting neatly within the unit circle.

This latter observation raises an interesting philosophical question. How can the points of a semi-infinite area (the right half of the $z$ plane) be mapped into a finite area within the $|\Gamma| \leq 1$ circle? The answer is similar to that of the question: “How many angels can dance on the head of a pin?” The answer is: “All the angels can, because angels do not require any space.” Similarly, impedance points, no matter how many of them, take up no area and therefore can be mapped from any area into any other area.

The solution to the two problems stated above lies in finding a means of mapping not individual points but contours from the $z$ plane to the $\Gamma$ plane. When we use the $z$ plane, we find the intersection of the $r = \text{const}$ and $x = \text{const}$ lines for the particular impedance at hand. Once these contours are mapped onto the $\Gamma$ plane (the Smith chart), we can enter the $\Gamma$ plane directly, using the normalized load impedance on a line; and, after the required clockwise rotation, can read directly the corresponding normalized input impedance at that point on the line. No complex calculations would be required to perform the input impedance calculation of (5.1-1).

In summary, Philip Smith’s first important idea in developing the Smith chart was recognizing that reflection coefficient rather than impedance should be used to track movement on a transmission line. The second important idea was the mapping of $r = \text{const}$ and $x = \text{const}$ contours in the $z$ plane to the $\Gamma$ plane. The $\Gamma$ plane with this mapping of normalized impedance is called the Smith chart.

Figure 5.1-3 Mapping discrete impedance points from normalized $z$ plane to $\Gamma$ plane under the transformation $\Gamma = (z - 1)/(z + 1)$. 

BASIS OF THE SMITH CHART

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5.2 DRAWING THE SMITH CHART

The normalized, passive impedance Smith chart is the polar plot of $|\Gamma| \leq 1$ with an overlay of $r = \text{const}$ and $x = \text{const}$ contours that relate $\Gamma$ to $z$. Normalizing the Smith chart allows its use with any characteristic impedance transmission line. Alternatively, one can draw a Smith chart for use with a given absolute value of $Z_0$, such as 50 $\Omega$.

The normalized, passive impedance Smith chart consists of constant resistance and reactance contours mapped from the $z$ to the $G$ plane according to the function

$$\Gamma = \frac{z - 1}{z + 1} \quad \text{where} \quad z = r + jx, \quad r \geq 0, \quad -\infty \leq x \leq +\infty \quad (5.2-1)$$

The mapping function of (5.2-1) is a bilinear transformation, which has four properties important to the drawing of the Smith chart [4]. These are:

1. The semi-infinite right half of the $z$ plane (for $r \geq 0$) is mapped into the $|\Gamma| \leq 1$ unit circle. (Note that the left half of the $z$ plane for which $r < 0$ would map into the semi-infinite area outside of the $|\Gamma| > 1$ circle, the negative resistance domain.)
2. The transformation maps circles into circles, with straight lines being considered circles of infinite radius and points being circles of zero radius.
3. The mapping is conformal (angle preserving). Contours in the $z$ plane that are orthogonal to each other will map into contours in the $\Gamma$ plane that likewise are orthogonal to each other.
4. The mapping is analytic. Therefore, continuous contours in the $z$ plane are mapped into continuous contours in the $\Gamma$ plane.

Usually the reflection coefficient $\Gamma$ is expressed in polar form as $\rho e^{i\varphi}$ or $\rho \angle \varphi$; however, it is difficult to recognize the equation of a circle in this format unless the circle’s center lies at the origin. Therefore, to demonstrate this mapping, we express $\Gamma$ in its Cartesian format, $\Gamma = e^{i\varphi} = u + jv$, retaining the same coordinate center for both polar and Cartesian representations.

The relationships between polar and rectangular coordinates are, from polar to rectangular

$$u = \rho \cos \varphi \quad (5.2-2a)$$
$$v = \rho \sin \varphi \quad (5.2-2b)$$

and from rectangular to polar

$$\rho = \sqrt{u^2 + v^2} \quad (5.2-3a)$$
$$\varphi = \tan^{-1}\left(\frac{v}{u}\right) \quad (5.2-3b)$$
Because a complex function such as $\Gamma = u + jv$ has two parts, real and imaginary, as does its complex independent variable, $z = r + jx$, it is not possible to graph $\Gamma(z)$ versus $z$ in a single plane as we would the real variable $f(x)$ versus $x$. Instead we will show how a contour in the $z$ plane, such as $r = 2$, is mapped into a corresponding contour in the $\Gamma$ plane.

In this mapping task, we wish to set $r = \text{const}$ for one set of contours and then $x = \text{const}$ for the other, orthogonal, set of contours to be drawn in the $\Gamma$ plane. To do this it proves convenient to use the relation that gives $z$ as a function of $\Gamma$. In this way $z$ can be set either to a constant $r$ or $jx$ and the expression solved for the $\Gamma$ rectangular coordinates. Thus,

$$z = \frac{1 + \Gamma}{1 - \Gamma} \quad (5.2-4)$$

where $z$ and $\Gamma$ continue to be functions of position $x$ on the transmission line (not to be confused with normalized reactance, $x$).

The relation (5.2-4) can be rewritten as

$$r + jx = \left(\frac{1 + pe^{j\varphi}}{1 - pe^{j\varphi}}\right) \left(\frac{1 - pe^{-j\varphi}}{1 - pe^{-j\varphi}}\right) \quad (5.2-5)$$

Carrying out the indicated multiplication to remove the complex quantities from the denominator and noting that $e^{j\varphi} - e^{-j\varphi} = 2j \sin \varphi$ and $e^{j\varphi} + e^{-j\varphi} = 2 \cos \varphi$ we obtain

$$r + jx = \frac{1 - p^2 + 2p \sin \varphi}{1 + p^2 - 2p \cos \varphi} = \frac{1 - u^2 - v^2 + 2ju}{1 + u^2 + v^2 - 2u} \quad (5.2-6)$$

This complex equation requires that separately both the real and imaginary parts of each side of the equation be equal to each other. Equating the real parts,

$$r = \frac{1 - u^2 - v^2}{1 + u^2 + v^2 - 2u} \quad (5.2-7)$$

We know from the properties of the bilinear transformation that (5.2-7) must be the equation of a circle in the $\Gamma = u + jv$ plane for $r = \text{const}$. But it requires some algebraic manipulation [5, p. 75] to cast (5.2-7) into the easily recognizable expression for a circle with center at $u_1 + jv_1$ and radius $a$, namely the format

$$(u - u_1)^2 + (v - v_1)^2 = a^2 \quad (5.2-8)$$

To do so, rewrite (5.2-7) as
The last form of \((5.2-9)\) is the recognizable expression for a circle in the \(\Gamma\) plane with center at

\[
u = \frac{r}{r+1} \quad j\nu = 0
\]

and radius equal to \(1/(r+1)\). For example, the contour \(r = 1\) in the \(z\) plane is mapped into a circle having radius 0.5 and center at \(u = 0.5, j\nu = 0\), as shown in Figure 5.2-1.

At this point in the mapping, we note that all impedance values having \(r = 1\) as a real part must lie on the \(r = 1\) circle in the \(\Gamma\) plane. We do not yet know how to place specific impedances, on this circle, since we have not yet developed the orthogonal \(x\) contours for the \(\Gamma\) plane. In a limited sense, we can deduce that \(\Gamma = 0\) when \(z = 1 + j0\), and that accordingly this impedance point must be the one at which the \(r = 1\) circle passes through the origin, since the
corresponding reflection coefficient magnitude must be zero. We can also say that at \( u = 1 \), \( jv = 0 \), \( \Gamma = +1 \) (an open circuit), and so this must correspond to the \( z = 1 \pm j\infty \) point.

As additional contours are plotted, the constant \( r \) lines in the \( z \) plane map into circles that are tangent to each other at the \( u = 1 \), \( jv = 0 \) point in the \( \Gamma \) plane, as shown in Figure 5.2-2.

Next, we map the orthogonal \( x = \text{const} \) contours into the \( \Gamma \) plane. To do so, equate the imaginary parts on both sides of (5.2-6) to obtain

\[
x = \frac{2v}{1 + u^2 + v^2 - 2u}
\]  

(5.2-10)

This is manipulated as

\[
\left(u^2 - 2u + 1 + v^2 - \frac{2v}{x} + \frac{1}{x^2} = \frac{1}{x^2}\right)
\]

\[
(u - 1)^2 + \left(v - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2
\]  

(5.2-11)

For \( x \) equal to a constant, this is the equation of a circle with center at \( u = 1 \), \( jv = j/x \) and radius equal to \( 1/x \). When contours are mapped for \( x = -2, -1, -0.5, 0, +0.5, 1, \) and 2, the results give partial circles within the \(|\Gamma| \leq 1\) unity circle of the \( \Gamma \) plane. The circles can be extended beyond this region, but the corresponding impedances have negative real parts, and our interest for

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**Figure 5.2-2** Constant-resistance lines of \( z \) plane map into circles tangent at \( u = 1 \), \( jv = 0 \) in \( \Gamma \) plane and centered on real axis.
now is the Smith chart for use with passive impedances only, that is, for reflection coefficients no greater than unity. More about negative resistance Smith charts later.

The contours in the $\Gamma$ plane shown in Figures 5.2-2 and 5.2-3 form orthogonal sets, mapping the $z = r + jx$ values into the reflection coefficient plane, as we desired. Notice that, while the contours are circular, at their intersections their lines are at right angles (orthogonal) to each other. When both sets of contours ($r = \text{const}$ and $x = \text{const}$) are combined, the result is the skeletal form of the Smith chart shown in Figure 5.2-4.

Notice that positive reactances (inductances) lie in the upper half and negative reactances (capacitances) in the lower half of the Smith chart (as was true in the $z$ plane).

The diagram of Figure 5.2-4 is very comprehensible because it shows the evolution of the Smith chart by presenting all of the necessary coordinates and scales. However, if the Smith chart is to be used graphically, as was the original intent, it must contain many more impedance contours in order that impedance can be read more precisely. For this reason the conventional format for the Smith chart is as shown in Figure 5.2-5.

Because so many $r$ and $x$ contours are required for a Smith chart that is to be used graphically to determine impedance transformations on transmission lines, there is insufficient space within the $|\Gamma| = 1$ circle to print explicit scales for the magnitude of reflection coefficient, even though reflection coefficient is the basis of the Smith chart. Also, the $j$ is not used ahead of reactive impedances or susceptive admittances.

That the Smith chart fundamentally is a plot of reflection coefficient is emphasized here because it is not uncommon for individuals to become quite
adept at using the Smith chart to transform impedances yet forget, or never have been aware of, the fact that the Smith chart is just a polar plot of reflection coefficient with impedance and/or admittance overlays.

Knowing this fact, it is easy to answer many other questions about transmission lines, such as what is the voltage magnitude at a given position $x$ on the line. It is simply $V_{1}|1 + \Gamma(x)|$ where $\Gamma$ is readily read for any specified point on the chart.

To facilitate the determination of reflection coefficient and other important transmission line parameters, there are radial scales printed at the bottom of the chart. These are used in combination with a pair of dividers to determine the magnitude of the reflection coefficient. Only the linear reflection coefficient magnitude scale, $\rho$, is necessary, since the angle of $\Gamma$ can be read from the degrees scale on the periphery of the Smith chart. The other parameters, return loss, mismatch loss, and other values are readily determined, given $\rho$; but for convenience separate radial scales for these values are also printed below the chart.

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Figure 5.2-4 Smith chart showing both impedance contours $r$ and $jx$ as well as $|\Gamma|$ contours and $\Gamma$'s angle $\varphi$ in $90^\circ$ segments.
5.3 ADMITTANCE ON THE SMITH CHART

At any location on the transmission line the reflection coefficient is related to the normalized impedance by

$$\Gamma = \frac{z - 1}{z + 1}$$

(5.3-1)
Since normalized admittance $y$ is just the reciprocal of normalized impedance $z$,

$$
\Gamma = \frac{1/y - 1}{1/y + 1} = \frac{1 - y}{1 + y}
$$

where the bilinear transformation in (5.3-2) is simply the negative of the transformation in (5.3-1). That is, the reflection coefficient plot as a function of $y$ is identical to that of $z$ except for a rotation within the $\Gamma$ plane of $180^\circ$. When the admittance coordinates for constant conductance $g$ and constant susceptance $b$ are plotted in the $\Gamma$ plane, the result is the admittance version of the Smith chart, shown in Figure 5.3-1.

This plot is identical to that of the impedance plot in Figure 5.2-4 except for a $180^\circ$ rotation about the center of the chart. All of the rules for transforming impedance along a line length apply equally to admittances. With normalized admittances, the rotation moving from the load toward the generator remains clockwise, as it was for impedances.

Figure 5.3-1  Smith chart with normalized admittance coordinates.
Inductors have positive reactance but negative susceptance. Capacitors have negative susceptance but positive susceptance with the result that the upper half of the Smith chart is always inductive and the lower half is always capacitive.

Due to the mirror symmetry of the impedance and admittance Smith charts, one could create an admittance Smith chart merely by rotating an impedance chart through 180°. In fact, for many years the practice was to consider that the normalized coordinates in Figure 5.2-5 could be defined to be either admittance or impedance, and the “standard” chart shown in Figure 5.2-5 was, and often currently is, labeled “impedance or admittance coordinates.”

The problem with defining these coordinates as admittance is that the angle of the reflection coefficient is in error by 180°, and this must be corrected when interpreting the “standard” Smith chart as an admittance Smith chart. Another problem is that this practice is more prone to errors, since it is easy to forget how the coordinates are presently defined, particularly since conversions between admittance and impedance frequently must be made in the solution of a problem.

The modern and preferred procedure is to print both normalized impedance and admittance contours on the same Smith chart, overlaying the coordinates derived in Figures 5.2-4 and 5.3-1, so that one can read directly at any given point either value without resort to crossing or rotating the chart. The resulting Smith chart, having both impedance and admittance contours, is shown in Figure 5.3-2.

To aid the graphical use of this chart, it is usually printed in two colors, for example, red for normalized impedance and blue for normalized admittance. Even so, as a graphical tool it is difficult to use. Fortunately, today there are software versions of the Smith chart for use on a personal computer. Fewer contours need be drawn because the computer allows entry onto the chart at the exact value of load impedance or admittance, creates the traces of transmission line and other transformations, and displays the exact value of input impedance or admittance, as will be seen. Furthermore, either impedance or admittance contours or both can be displayed as desired. This will be demonstrated in the examples to follow.

5.4 TUNING A MISMATCHED LOAD

Originally, the Smith chart was developed to facilitate the determination of load mismatch and the requisite tuning that would match it to the characteristic impedance of the transmission line. While the chart has other useful applications, this load tuning remains a principal application of the Smith chart.

Movement along a lossless line corresponds to a contour having a constant reflection coefficient magnitude, \( \rho \), and thus a constant radial distance from the center of the Smith chart, that is, a circle with center at the center of the chart. The angle, \( \phi \), of the reflection coefficient, \( \Gamma = \rho e^{j\phi} = \rho \angle \phi \), changes at twice the rate of movement along the line because it is the ratio of an incident and a
reflected wave, each of whose arguments change by an amount equal to the movement and their angles are additive in taking their ratio. Movement from the load toward the generator on the Smith chart is clockwise.

To facilitate the computation of movements, the Smith chart has three peripheral scales. The innermost scale gives degrees of the angle of $\Gamma$, a total of $360^\circ$ for the full circle. The outer two scales are labeled “wavelengths toward the generator” and “wavelengths toward the load.” These two scales go through one-half wavelength for the full circle, thereby taking into account that the reflection angle changes by twice the angle of movement along the transmission line. This movement is demonstrated in the following example.
For this example, we return to the case of Figure 4.14-5 consisting of a $(25 + j30)\,\Omega$ load at the end of a $50^\circ$ long section of $50\,\Omega$ transmission line. Normalizing the load impedance to $50\,\Omega$, it becomes $0.5 + j0.6$. This is shown as point $A$ in Figure 5.4-1.

To determine the arc length movement on the Smith chart, we note that traveling $50^\circ$ along the line toward the generator results in a reflection coefficient angle change of $-100^\circ$ (clockwise rotation) to point $B$, at which we read a normalized impedance of $2.8 + j0.5$. Unnormalized (multiplying both parts by $50\,\Omega$) this is $(140 + j25)\,\Omega$, reasonably close to the value of $139 + j24$ calculated in (4.14-15), given the graphical accuracy of the Smith chart, an analog instrument. Note the economy of effort with which this result is obtained relative to the solution with the input impedance formula.

As an alternative to the determination of the arc length movement on the Smith chart, we can use the “wavelengths toward generator” scale. Traveling $50^\circ$ on the line toward the generator results in a clockwise movement on
the chart by $50^\circ/360^\circ = 0.139$ wavelengths. This dimension is shown in Figure 5.4-1 and results in the same travel from point $A$ to point $B$.

From Figure 5.4-1 the value of the Smith chart is evident. The extended circular contour (shown dashed) at a constant radial distance from the center of the Smith chart reveals all of the normalized impedance values that would be encountered if the transmission line were extended to a half wavelength or longer.

Specifically, we see that, if the transmission line is lengthened until the reflection coefficient angle is $-61^\circ$ (point $C$) the normalized input impedance will be $1.0 - j1.1$ ($50 \Omega - j55 \Omega$ when unnormalized). At point $C$ the total change in reflection coefficient angle measured from point $A$ is $-169^\circ$, corresponding to a total transmission line length of $84.5^\circ$. To reach point $C$ we must add an additional $34.5^\circ$ of 50-\Omega line, corresponding to a clockwise movement on the chart of an additional 0.096 wavelengths. At point $C$ the insertion of an inductive reactance of 1.1 ($55 \Omega$ unnormalized) in series with the line cancels the capacitive reactance and moves the normalized input impedance to 1 ($50 \Omega$ unnormalized) at point $D$, thereby matching the load to the transmission line.

This example demonstrates that not only does the Smith chart facilitate the calculation of the input impedance to a line terminated in a mismatched load, it also readily reveals how the mismatch can be tuned to yield a matched input.

This tuning method is not unique. Actually, there are an infinite number of possible tuning approaches that can be determined with the skillful use of the Smith chart. By continuing clockwise on the dotted circular arc, an additional 0.27 wavelength, between points $C$ and $A$, respectively, of Figure 5.4-1, we can observe the remainder of the wide range of normalized impedance values that would have been encountered on the transmission line had the electrical length, $\theta$, been varied from $0^\circ$ to $180^\circ$. The total excursion is 0.5 wavelength, after which we are returned to the starting impedance. This duplicate of the load impedance occurs every half wavelength multiple from the load along a lossless line, as was noted in Section 4.14.

In the current example, the real part of the normalized input impedance varies between 0.33 and 3, a 9-to-1 ratio. The imaginary part varies between $-j1.2$ and $+j1.2$. Using only the input impedance formula, these conclusions, obvious with the Smith chart, would have been very laborious to reach. For this reason the Smith chart is a universal presentational format for microwave measurements, usually a direct data output option for network analyzer measurements and network simulation software.

### 5.5 SLOTTED-LINE IMPEDANCE MEASUREMENT

From the 1930s to the 1950s, before network analyzers were available, the slotted line was used to measure microwave impedance. Even today, VSWR, the data value obtained with a slotted line, continues to be a common term used to describe the tolerance of impedance matching.
The slotted-line measurement was effected by cutting a nonradiating slot along the direction of propagation in a coaxial or waveguide transmission line and sliding a rectifying detector diode along the slot to sample the rectified time-average voltage amplitude as a function of position along the line. This fixture was called a slotted line. The ratio of the maximum to minimum voltage values is the voltage standing-wave ratio, or VSWR. The VSWR coupled with the location of the associated voltage minimum (null) could be used to determine the load impedance $Z_L$. The following example demonstrates this procedure.

First, a short circuit is connected at the end of the transmission line prior to connecting the load (Fig. 5.5-1a), and the positions of the resulting voltage minima, or nulls, recorded. These null locations are equivalent to the electrical position of the load plane. The rectified voltage amplitude varies sharply with position near a null (Fig. 5.5-1a), yielding the most precise measurement of the equivalent load positions. The distance between two successive minima is one-half wavelength at the operating frequency, providing a means of determining the actual operating frequency of the generator. Each set of VSWR and null data is taken at a single microwave frequency. To obtain data for a band of frequencies, the measurement must be re-performed at each frequency of evaluation.

For example, suppose that, with the short circuit installed, nulls are found at 2.495, and 7.90 in. along the slotted line. The distance between minima is a half wavelength, therefore

$$\lambda = 2(4.95 \text{ in.} - 2.00 \text{ in.}) = 5.90 \text{ in.} \quad (5.5-1)$$

If the slotted line has air dielectric, as is usually the case to permit unimpeded probe movement,

$$f_0 = \frac{11.8 \text{ in.} \cdot 1000 \text{ MHz}}{5.9 \text{ in.}} = 2000 \text{ MHz} \quad (5.5-2)$$

This demonstrates that the slotted line provides a means of frequency determination.

The locations of the voltage minima can be considered equivalent to the electrical position of the load plane, repeating at half wavelength intervals.

Next, the load is attached and the rectified voltage measurements yield $V_{\text{MAX}} = 1.5$ units and $V_{\text{MIN}} = 0.5$ units (the actual voltage scale is unimportant) with null positions at 2.59 and 5.54 in. From these data

$$\text{VSWR} = \frac{V_{\text{MAX}}}{V_{\text{MIN}}} = \frac{1.5}{0.5} = 3 \quad (5.5-3)$$

$$\rho = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} = \frac{3 - 1}{3 + 1} = 0.5 \quad (5.5-4)$$

The locations of the voltage minima can be considered equivalent to the electrical position of the load plane, repeating at half wavelength intervals.

Next, the load is attached and the rectified voltage measurements yield $V_{\text{MAX}} = 1.5$ units and $V_{\text{MIN}} = 0.5$ units (the actual voltage scale is unimportant) with null positions at 2.59 and 5.54 in. From these data

$$\text{VSWR} = \frac{V_{\text{MAX}}}{V_{\text{MIN}}} = \frac{1.5}{0.5} = 3 \quad (5.5-3)$$

$$\rho = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} = \frac{3 - 1}{3 + 1} = 0.5 \quad (5.5-4)$$
The null with load in place is located a distance \( D = 7.90 \text{ in.} - 5.54 \text{ in.} = 2.36 \text{ in.} \) toward the generator from the load plane. To locate the normalized load impedance we convert this distance to wavelengths.

\[
D = \frac{2.36 \text{ in.}}{5.90 \text{ in./wavelength}} = 0.40 \text{ wavelength}
\]

\((5.5-5)\)
On the Smith chart a half wavelength of travel on the transmission line produces a $360^\circ$ change in the argument of the reflection coefficient:

$$0.5\lambda \text{ of travel on line } \Rightarrow 360^\circ \text{ around chart} \quad (5.5-6)$$

Therefore, in degrees

$$D = 0.4\lambda \left( \frac{360^\circ}{0.5\lambda} \right) = 288^\circ \quad (5.5-7)$$

The circumferential scales on the Smith chart read both in degrees of reflection coefficient angle and in wavelengths. Starting from the short-circuit point $(z = 0, \varphi = 180^\circ)$ the distance $D$ is shown plotted in the Smith chart of Figure 5.5-2. Note that the $288^\circ (0.4\lambda)$ arc is drawn counterclockwise toward the load.
since this is the direction from the load minimum to the plane of the load in question.

The angle of the reflection coefficient at the load is read directly from the peripheral scale on the Smith chart as

$$\varphi = 108^\circ$$  \hspace{1cm} (5.5-8)

and hence

$$\Gamma_L = \rho \angle \varphi = 0.5 \angle 108^\circ$$  \hspace{1cm} (5.5-9)

At this point the normalized load impedance can be read directly from the chart or calculated from

$$z_L = \frac{1 + \Gamma_L}{1 - \Gamma_L} = \frac{1 + 0.5 \angle 108^\circ}{1 - 0.5 \angle 108^\circ} = 0.5 + j0.6$$ \hspace{1cm} (5.5-10)

Unnormalized this is \((25 + j30) \, \Omega\), the value used in the example of Figure 4.14-5.

Often the difficult part of the slotted-line measurement is deciding in which direction to rotate on the Smith chart from the null position obtained with the load in place. It may help to remember that if the load impedance were known and plotted on the Smith chart, one would make a negative angle rotation (clockwise from the load toward the generator) to the position of the minimum impedance (the \(\pm 180^\circ\) axis of the chart). Therefore, to travel from the null toward the load rotate counterclockwise on the Smith chart.

The reader will notice from this example, in which the VSWR is 3, that the constant reflection coefficient circle intersects the \(r = 3\) circle on the \(0^\circ\) axis of the Smith chart. The circle always intersects the \(r\) circle at the value of VSWR, as will be demonstrated in the next section.

### 5.6 VSWR = \(r\)

When a constant magnitude reflection coefficient circle of radius greater than zero is drawn on the Smith chart, it intersects the horizontal axis of the chart at two points, \(r\) and \(1/r\). In the case of a matched load, \(\rho = 0\), the circle has zero radius, and the two intersections coalesce into one point, the center of the chart at which \(r = 1, \varphi = 0^\circ\).

For the intersection for which \(r > 1, z = r + j0\), the reflection coefficient has a real value given by

$$\Gamma = \frac{z - 1}{z + 1} = \rho = \frac{r - 1}{r + 1}$$ \hspace{1cm} (5.6-1)
The corresponding VSWR is

\[
VSWR = \frac{1 + \rho}{1 - \rho} = \frac{r + 1 + r - 1}{r + 1 - r + 1} = r
\]  

(5.6-2)

For this reason the VSWR for any normalized load impedance can be obtained by drawing the constant \( \rho \) circle through it. The VSWR is numerically equal to the resistance value \( r \), corresponding to the circle’s intercept with the horizontal Smith chart axis for which \( r > 1 \).

5.7 NEGATIVE RESISTANCE SMITH CHART

The area that is outside the unit circle of the Smith chart corresponds to reflection coefficient magnitudes greater than one, the negative resistance and conductance domain. The resistance circles defined by (5.2-9) can be plotted for negative values of \( r \), and the circles for constant reactance can be extended outside the \( \rho = 1 \) circle. For \( \rho \leq 3.16 \) (a reflection gain of 10 dB) the result is that shown in Figure 5.7-1. The example of impedance \( Z_{IN}/Z_0 \) and admittance \( Y_{IN}/Y_0 \) shown on this standard chart is consistent with identifying the admittance of a point on a 180° diametrically opposite location. However, as was noted earlier, the chart can and should be drawn with simultaneously valid \( Z \) and \( Y \) coordinates in the format of Figure 5.3-2.

Unlike the passive impedance/admittance domain of the Smith chart, the negative resistance domain is unbounded. A new chart must be drawn for each maximum anticipated value of \( \rho \).

The negative resistance domain of the Smith chart (where \( \rho > 1 \)) is defined by the same equations used for the passive Smith chart. This negative resistance domain is useful, for example, to describe reflection amplifiers and to plot the stability circles for transistor amplifiers. However, for most matching designs, the area within the passive impedance Smith chart is sufficient and our matching examples are confined to this area.

5.8 NAVIGATING THE SMITH CHART

For lossless matching, two principal paths, or contours, on the Smith chart are employed. These are

1. Constant radius contours for movement along a transmission line and
2. Constant resistance (or conductance) contours for the addition of reactance (or susceptance)

Movement on constant reactance (or susceptance) circles is also possible but implies the addition of resistance (or conductance), consequently lossy match-
Lossy matching has its place, as for very broadband matching, but it is usually avoided in favor of lossless matching.

If the load on a transmission line contains no resistance or conductance, (1) the load can absorb no power, (2) the reflected and incident waves have the same magnitude, (3) the reflection coefficient magnitude is unity ($|\Gamma| = 1$), and (4) $\Gamma$ traces the rim of the passive Smith chart, traversing every reactance (and susceptance) from $-j\infty$ to $+j\infty$ in a transmission line travel of one-half wavelength. This means that open- or short-circuited transmission lines, or stubs, can be used as reactive/susceptive tuning elements.

The next example demonstrates these Smith chart contours and stub matching. A 100-Ω load is connected in series with a 1-pF capacitor. The resulting impedance at 1 GHz of $(100 - j159)$  Ω is to be connected to the end of a 50-Ω transmission line using wire leads having a total of 10 nH of inductance.
The resulting load is to be matched to 50 Ω using a shunt tuning stub. The stub may be either open or short circuit terminated. The type of stub termination, the length of the stub, and its location on the main line is to be determined to match this load to 50 Ω (Fig. 5.8-1).

The matching is carried out in the following steps:

1. The reference impedance of the Smith chart is defined to be 50 Ω (the chart is normalized to 50 Ω).
2. The load impedance \((100 - j159) \Omega\) is divided by 50 \(\Omega\) to normalize it, giving \(2 - j3.2\) (to the accuracy consistent with the graphical process). This is point \(A\) in Figure 4.8-1.

3. The \(+j63-\Omega\) series inductance has a normalized value of \(+j1.3\). This is added to the load impedance by traveling along the constant resistance circle \((r = 2)\) to the reactance intercept \(-j1.9\). The initial load (point \(A\)) is in the lower half of the Smith chart since its reactance is capacitive. Adding the series inductance causes an upward movement, toward the inductive upper half of the chart, to point \(B\).

4. Drawing a clockwise, constant reflection coefficient magnitude arc intersects the unity conductance circle at point \(C\). This point is important. We wish to add a shunt stub. The shunt addition requires the use of admittance coordinates, hence intercept on the unity conductance circle.

Using the wavelength scale of the chart’s periphery, we determine that the starting point \(B\) is at 0.292\(\lambda\), and that the \(g = 1\) intersection, point \(C\), is at 0.425\(\lambda\). The distance traveled is 0.425\(\lambda\) - 0.292\(\lambda\) = 0.133\(\lambda\), or 48°. This is the distance from the load at which the shunt tuning stub is to be connected. The length of line between load and stub is 48°. This is not to be confused with the angular change in \(\Gamma\), which is \(-2(48°) = -96°\).

5. At point \(C\) the normalized admittance is read from the chart to be \(1 + j1.5\). To obtain a match, add a normalized susceptance of \(-1.5\) thereby traveling on the \(g = 1\) circle to the center of the chart, point \(D\), completing the match.

6. Short-circuited or open-circuited transmission lines can be used as tuning elements of any desired reactance. We have previously seen that such stubs are described by (4.14-7a) and (4.14-8a), but the Smith chart allows their reactances or susceptances to be read directly. We are free to use any characteristic impedance stub. Suppose for this example that we choose \(Z_0 = 50 \Omega\).

Then the same Smith chart normalization applies, and we can perform this calculation on the chart used to determine the matching requirement. To obtain the required \(-j1.5\) susceptance (an inductance) we begin with a shorted stub at the \(y = \infty\) location (point \(E\)) on the Smith chart and travel clockwise along the chart’s periphery to the required inductive susceptance at point \(F\). This requires a stub length of 0.0955\(\lambda\), or about 34.4°. It is emphasized that this is a side calculation and this travel is not part of the original contour from \(A\) to \(D\).

Shorted stubs are inductive when their length is less than a quarter wavelength. An open-circuited, 50-\(\Omega\) stub could be used, but its length would be a quarter wavelength longer (0.3455\(\lambda\) instead of 0.0955\(\lambda\)). Consequently, its frequency sensitivity would be nearly four times greater than that of the shorted stub.
Finally, the same Smith chart can be used even if the tuning stub impedance is different from 50 Ω. For example, suppose a 100-Ω stub is used, then

\[ Y_{STUB} = -j1.5(0.02 \, \Omega) = -j0.03 \, \Omega \]

\[ Y_{STUB} \text{ (normalized to 100 } \Omega) = \frac{-j0.03}{0.01} = -j3 \]

This is point G in Figure 5.8-1 and corresponds to a stub length of about 0.05λ. Notice that for this side calculation the Smith chart is normalized to 100 Ω. This is a shorter stub and would take up much less space in the circuit layout, both because it is half the length of the 50-Ω stub and because 100-Ω stubs have much narrower width, hence can be “serpentined” more easily. Shorter stubs also can have less frequency variation, however, both the 50- and 100-Ω stubs of this example are so short that their susceptances would not vary much more rapidly than inversely with frequency.

The completed circuit with 50-Ω stub is shown in Figure 5.8-2 along with its VSWR performance, referenced to 50 Ω. Errors in round off and graphical inaccuracies make the result slightly off of a perfect 1.00 VSWR and the frequency of best match slightly above the design value of 1000 MHz. This result greatly improves the efficiency of power transmission to the load, as indicated below.

When the arc described by points B and C is extended to the 0° axis of the Smith chart, it intersects it at about \( r = 4.2 \). Therefore, the VSWR of the load,
including its series inductance, is 4.2. The reflection coefficient $\rho$ is 0.62. This is read using the auxiliary scale (not shown in the figure) at the bottom of the standard Smith chart sheet or calculated from VSWR and (4.5-3b).

Without the stub tuning, the return loss (fraction of power returned to the generator) of the load before tuning would have been

$$RL = \rho^2 \times 100\% = 38\% \quad (5.8-1)$$

Thus the mismatch loss is 62%, which corresponds to a mismatch loss of 2.1 dB. With the stub tuning, the VSWR is below 1.1 and $\rho$ is below 0.05 from (990 – 1020) MHz. The corresponding return loss is less than 0.3%. This improvement in efficiency, easily designed using the Smith chart, is obtained merely by printing a side stub on the transmission line and grounding it.

### 5.9 SMITH CHART SOFTWARE

Although the procedures for using the Smith chart as a graphical tool are fairly simple, it is easy to err by rotating in the wrong direction, misreading the coordinate values or trying to match using the incorrect circle. To minimize errors, remember to: *Rotate to the $r = 1$ circle when adding a series reactance* and *rotate to the $g = 1$ circle when adding a shunt susceptance*.

Furthermore, the accuracy of the procedure is limited by the graphical process itself. Finally, the Smith chart with both impedance and admittance coordinates is a crowded format that is difficult to read.

These problems are considerably lessened through the use of *Smith chart software* written for the personal computer. The following examples are illustrated using one such program called winSmith [6].

For example, given a load of $(25 + j30) \Omega$ at 2000 MHz terminating a 50-$\Omega$ transmission line, it is desired to match this load to 50 $\Omega$ using a tuning element connected in shunt with the transmission line at an appropriate distance from the load on the generator side. What is the electrical spacing, element type, and value for this tuning element?

This load point is entered as $(25 + j30) \Omega$ directly into the program along with the reference impedance for the chart. The program automatically locates impedances on the chart given the reference impedance so there is no need for normalization by the user. The parameters of the cursor-selected point (x in Fig. 5.9-1) are listed in the upper right corner of the display. The units of $Z$ and $Y$ are ohms and mhos, respectively. The $S$ term is $S_{11}$ with magnitude in decibels and angle in degrees ($S$ parameters are discussed in 6.4). The term $G$ stands for gamma ($\Gamma$), the reflection coefficient, listed here as $0.49 \angle 108^\circ$. This is a more precise value than the $0.5 \angle 108^\circ$ value that we read graphically from the Smith chart in the earlier example. The corresponding VSWR, listed as “$V$” in the display, is 2.89, again a more accurate value than the 3.0 value previously read graphically from the chart.
A selection of pull-down components in the upper left of the display allows the construction of a cascade circuit layout of various topologies. The resulting Smith chart contour is automatically computed and displayed for the circuit chosen and the variable values entered.

Both impedance (dark lines) and admittance (light lines) coordinates are shown. Recall that the location of the load is the same using this Smith chart format, whether the load is expressed in impedance or admittance coordinates. Unlike the graphical Smith chart, the software displayed Smith chart is not normalized, since the value of $Z_0$ is also entered. The impedance and admitance are read directly in ohms and mhos, respectively.

The tuning procedure is simplified further by selecting for display only the desired coordinates, impedance, or admittance appropriate to the current movement being made on the chart. We wish to move along the 50-Ω line, a constant reflection coefficient magnitude, from the load (now relabeled point $A$ in Fig. 5.9-2) to the intersection with the 0.02-Ω conductance circle (point $B$). Here the addition of an appropriate shunt susceptance will match the input to the line to 50 Ω (0.02 Ω). The admittance at point $B$ lies in the lower half of the Smith chart, and therefore its imaginary part corresponds to a capacitor (positive susceptance). It is essential to keep in mind that the lower half of the Smith chart...
chart is capacitive, and the upper half is inductive, whether expressed as impedance or admittance.

From the Smith chart, the 50-W line length required to move from point A to the unity conductance circle (point B) is 114° (0.32λ). Notice that this results in a clockwise rotation of −228° on the Smith chart. From the cursor reading (not visible in this view), the admittance at point B was read to be (0.02 + j0.022) Ω. The addition of a 3.54-nH shunt inductor (having susceptance of −j0.022 Ω at 2 GHz) moves the contour to point C, tuning the load to 50 Ω, as shown in Figure 5.9-2.

5.10 ESTIMATING BANDWIDTH ON THE SMITH CHART

Each contour on the Smith chart applies at a single frequency. However, a measure of the bandwidth of a particular tuning procedure can be obtained by adding a contour at each band edge. Suppose that we are interested in the bandwidth 1800 to 2200 MHz for the previous example, shown plotted in Figure 5.9-2. Adding these band edge contours yields the results in Figure 5.10-1. The load has been modeled as a 25-Ω resistor in series with a 2.39-nH inductor (+j30 Ω at 2000 MHz) in order to simulate the frequency variation of the load.
Notice that the contours for 1800 MHz (terminating at point \( D \)) and 2200 MHz (terminating at point \( E \)) show large variations in the angle for the transmission line section. This is because the line length changes by \( \pm 11.4^\circ \) at these band edges. This change is further amplified because on the Smith chart the angular change is doubled to \( \pm 22.8^\circ \). Using the cursor, the VSWR values at the end of these excursions were 1.94 at 1800 MHz and 1.96 at 2200 MHz. The resulting mismatch loss is about 0.5 dB.

### 5.11 APPROXIMATE TUNING MAY BE BETTER

The engineer who first learns matching with the Smith chart is inclined to try for a “perfect match” at the design center frequency, on the assumption that this will also give the best performance over a given bandwidth. But this assumption is not necessarily valid. In some cases an approximate solution at the center frequency may give better overall performance. In addition, settling for a reasonably “close match” at the center frequency might be obtainable using more economical tuning elements, while actually providing a better overall match throughout the required frequency bandwidth. The following example illustrates such a case.

Referring to Figure 5.9-1, we see that the \( 25 + j30 \) \( \Omega \) starting impedance (point \( A \)) lies close to but is not on the unity conductance circle. At this point
the admittance is $(0.0164 - j0.197)$ Ω. If a capacitor is added in shunt with the load (point A) having a susceptance of $+j0.0197$ Ω ($-j50.8$ Ω) at 2000 MHz, it must have a capacitance of 1.56 pF. The resulting total admittance at point A would be $0.0164$ Ω (61 Ω real), corresponding to a 1.22 VSWR in a 50-Ω system. With this approximate tuning at the center frequency, the three frequency contours are as shown in Figure 5.11-1.

With this tuning the VSWR values are 1.15, 1.22, and 1.42 at 1800, 2000, and 2200 MHz, respectively. The highest mismatch loss in the band, at 2200 MHz, is only 0.13 dB, much better than was obtained when the load was perfectly matched at 2000 MHz using the transmission line and shunt inductor.

This example further exemplifies the advantage of the Smith chart. The opportunity for approximate tuning was evident when the load was shown with admittance coordinates on the Smith chart. The example demonstrates two points about matching.

First, *a broader band match usually can be obtained when the tuning is performed close to the load*. Second, *accepting an approximate match at the center frequency may result in a better average match over the operating band.*

Figure 5.11-1  Result of tuning the $(25 + j30)$-Ω load in an approximate manner at $f_0$ using a parallel capacitor.
5.12 FREQUENCY CONTOURS ON THE SMITH CHART

If the points in Figures 5.10-1 and 5.11-1 that describe low to high frequency impedances for the same location in the circuit are connected with a line, they can be seen to describe a clockwise arc. This is always the case, a consequence of Foster’s reactance theorem.

Foster’s reactance theorem states that the reactive portion of impedances and the susceptive portion of admittances increase positively as frequency increases.

Impedances and admittances on the Smith chart trace clockwise arcs as frequency is increased.

Often the swept frequency response of a network analyzer is presented on a Smith chart display without frequency labels. The clockwise increase with frequency is a ready means of identifying the low and high frequency ends of the display trace.

5.13 USING THE SMITH CHART WITHOUT TRANSMISSION LINES

Notice that in the preceding example there were no transformations along transmission lines (constant reflection coefficient magnitude arcs). The VSWR values were calculated as those which would be incurred if the load and its tuner were connected to a 50-Ω line.

The Smith chart can be used without transmission lines, even though it was developed as an aid to transforming impedances along them. It is an orthogonal set of resistance and reactance contours (or conductance and susceptance contours). It is especially useful for performing matching calculations because every passive impedance or admittance, including those with infinite magnitudes, can be plotted within the unit circle enclosed by the \( |\Gamma| \leq 1 \) Smith chart. This is not true of a rectangular impedance or admittance chart for any given scale factor, whose dimensions increase without bound with the magnitudes of the impedance and admittance components to be plotted.

When using the Smith chart without an associated transmission line, we are free to select arbitrarily its reference impedance, the \( Z_0 \) impedance to which all others are normalized. For the best graphical accuracy it is desirable to select the reference impedance near the geometric mean between the largest and smallest impedance values to be plotted.

For example, consider the \( Q \) matching example of Figure 3.5-2. We desired to \( Q \) match a 5-Ω resistor to 50 Ω at 1 GHz. To demonstrate how this matching could have been performed using the Smith chart instead of the \( Q \) matching method, we start by selecting the reference impedance of the chart arbitrarily to be 25 Ω. The 5 Ω resistance is first plotted on the Smith chart (point \( A \)) and a value of series inductance added to bring the contour to the intersection with the \( G = 0.02 \) conductance circle (point \( B \)), as shown in Figure 5.13-1.

At this point the cursor indicates a susceptance of \(-j0.06 \) Ω. In reactance this is \( +j16.7 \) Ω, which is resonated at 1 GHz using a capacitor of 9.54 pF.
When this value is added, the contour extends to point $C$, the $R = 50 \, \Omega$ value on the $25 \, \Omega$ reference impedance Smith chart. These tuning elements are the same as those derived earlier using the $Q$ matching procedure (see Figure 3.5-2).

5.14 CONSTANT $Q$ CIRCLES

In a Cartesian-type impedance plane the contours of constant $Q$ are defined by the straight lines $X = QR$, where $X$ is the reactance and $R$ the resistance of a series impedance. We saw that the Smith chart is produced by a bilateral transformation having the property that circles are transformed into circles, straight lines being cases of circles of infinite radius. Thus we expect that constant $Q$ lines drawn in the Smith chart ($\Gamma$) plane will be circular or straight lines. This is the case. When the $Q = 3$ circles are drawn in the Smith chart of the previous example, the result is that shown in Figure 5.14-1.

Recall that $Q = 3$ as a condition of the $Q$ matching procedure used in Figure 3.5-2. Given this $Q$ value, the 5- to 50-Ω transformation could be performed on the Smith chart by finding the intersections with the $Q = 3$ circles.

When $Q$ matching is used, consisting of reactances, which alternate in type ($L$ or $C$) and in topology (series/parallel), adhering to a low $Q$ value ensures broad bandwidth. When viewed on the Smith chart the matching contour...
remains close to the horizontal ($X = 0$) axis for low $Q$ values. In the broadband matching example of Figure 3.5-5, the $Q$ was reduced to 1.074. When this example is plotted on the Smith chart, the result is that shown in Figure 5.14-2.

Notice that the $Q$ circles are symmetric with respect to the $x = 0$ axis of the Smith chart. This suggests that the dual of this matching circuit, consisting of a series capacitor connected to the 5-$\Omega$ load, followed by a shunt inductor and alternating $C$ and $L$ elements thereafter would produce the same result. This is the case, as shown in Figure 5.14-3.

The element values shown in Figure 5.14-3 were found graphically by extending the contour segment for each tuning element addition to intersect alternately the $Q = 1.074$ circle and the horizontal axis. As such, they are
approximate. The final circuit obtained using these approximate values transforms 5 to 49.5 \( \Omega \). With more careful graphical techniques, a closer approximation to 50 \( \Omega \) could be obtained, but the associated VSWR of 1.01 of this result would be sufficient for practically all applications.

5.15 TRANSMISSION LINE LUMPED CIRCUIT EQUIVALENT

Often the properties of a uniform transmission line section are desired but space does not permit the use of an actual section of line. An example is the realization of the branch line hybrid coupler (covered in Section 8.9) using lumped elements. In such cases a lumped equivalent circuit for a transmission line section is useful. The equivalent circuit is only equivalent at one frequency, but the approximation over a modest bandwidth is adequate for many applications.

As an example, suppose that we wish to simulate a 90° length of 50-\( \Omega \) line using an \( LCL \) tee circuit at 1 GHz (Fig. 5.15-1). What are the element values for the equivalent circuit?

---

**Figure 5.14-3** Dual of broadband tuning circuit derived graphically using lower \( Q = 1.074 \) circle.
We recognize that any length of 50-Ω transmission line presents an input impedance of 50 Ω when the line is terminated in 50 Ω. After some experimentation on the Smith chart, we find that, starting with a load of 50 Ω (and normalizing the chart to this value), there are any number of \( L \) and \( C \) reactance combinations that will yield 50 Ω input impedance. A set of five combinations is shown in Table 5.15-1.

It might seem that this Smith chart application has failed since it does not provide a unique circuit solution, but this is not so. We have only required that the circuit deliver 50 Ω to the input when the load is 50 Ω, but any 50-Ω line length does so; and the Smith chart results properly indicate that numerous equivalent circuits exist for line sections of various lengths. To determine the

### Table 5.15-1 Sets of \( L \) and \( C \) Values Yielding 50 Ω Input to Tee Circuit

<table>
<thead>
<tr>
<th>Combination</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_L )</td>
<td>25</td>
<td>35</td>
<td>50</td>
<td>75</td>
<td>100</td>
</tr>
<tr>
<td>( X_C )</td>
<td>−63</td>
<td>−53</td>
<td>−50</td>
<td>−54.1</td>
<td>−62.6</td>
</tr>
<tr>
<td>( L )</td>
<td>3.98 nH</td>
<td>5.57 nH</td>
<td>7.96 nH</td>
<td>11.94 nH</td>
<td>15.92 nH</td>
</tr>
<tr>
<td>( C )</td>
<td>2.52 pF</td>
<td>3.0 pF</td>
<td>3.18 pF</td>
<td>2.94 pF</td>
<td>2.54 pF</td>
</tr>
</tbody>
</table>
equivalent circuit specifically for a 90° long, 50-Ω line section, we must add another condition that relates input and output impedances unique to a 90° long, 50-Ω line section. One such combination is a short-circuit load, which is transformed to an open circuit by a 90° line section as indicated by (4.14-7a). Among the test cases of Table 5.15-1, we find that case 3 yields this result, as shown in Figure 5.15-2.

Initially, we locate the short circuit at point A. Adding a series reactance of +50 Ω takes the contour to point B. Note that this is the $z = 0 + j1$ point on the chart normalized to 50 Ω. Next we must use admittance since the capacitor of the tee circuit is a shunt element. However, the contour remains on the periphery of the Smith chart because both the $r = 0$ and the $g = 0$ circles lie on the $|\Gamma| = 1$ circle. To reach the horizontal axis we must add a normalized capacitive susceptance of +1. Notice that this brings us to the infinite impedance, point C, on the Smith chart, regardless of the value of the remaining series inductor. However, this second inductor is necessary since only the cases listed in Table 5.15-1 give a matched 50-Ω input when the circuit is loaded with 50 Ω. Given this result, it is now clear why the capacitive and inductive reactances must be equal; they must be parallel resonant in order that the input
will be an open circuit when the output of the tee circuit is short circuited. While this is now obvious, the Smith chart revealed this fact graphically.

As a test of this equivalent circuit, consider another pair of load and input impedances. We noted earlier that a quarter-wave transmission line behaves as an impedance inverter. For example, when terminated in \(25\) W, the \(90^\circ\) section of a \(50\)-\(\Omega\) line presents \(100\) W at its input terminals. The tee circuit must provide the same transformation if it is to be equivalent.

We begin at point \(A\) (Fig. 5.15-3), which represents the \(25\)-\(\Omega\) load on a Smith chart normalized to \(50\) \(\Omega\). Adding a series inductor follows a constant resistance circle to point \(B\). Adding a shunt capacitor follows a constant conductance circle to point \(C\). Then adding a series inductor of the same value as used between points \(A\) and \(B\) follows a constant resistance contour to point \(D\), the \(100\) \(\Omega\) required input resistance value.

This three-element tee equivalent of a transmission line section can be analyzed exactly using the \(ABCD\) matrix approach of the next chapter. However, the matrix solution is more difficult to apply for a five-element equivalent circuit, as might be employed to achieve a broader bandwidth of operation.

![Figure 5.15-3](image-url) The circuit and Smith chart contour of the three-element, tee section equivalent circuit of a \(90^\circ\) long, \(50\)-\(\Omega\) transmission line at 1 GHz.
The five-element model was designed using the Smith chart, maintaining circuit symmetry (end two elements are equal to each other, as are the second and fourth elements) and requiring that a 25-Ω load result in a 100-Ω input. After another cut-and-try process on the Smith chart, the symmetric five-element circuit and its contour shown in Figure 5.15-4 was obtained. The results are approximate, but as will be seen, the match is quite close to 1.00 at the center frequency.

Notice that the contour is just inside a \( Q = 1.2 \) circle drawn on the chart, while the contour for the three-element circuit is just inside the \( Q = 2 \) circle. Due to the lower \( Q \) of the five-element circuit we would expect broader bandwidth for it, and this is the case as can be seen in Figure 5.15-5. For these simulations the input VSWR is referenced to 100 Ω in order to indicate the deviation of \( Z_{IN} \) from the required 100-Ω value.

These examples show that the Smith chart can be employed to design matching networks having considerable complexity, which might otherwise make them more difficult to design by purely analytical means. The examples
also are intended to demonstrate that what can be obtained using the Smith chart is only limited by the imagination and skill of its user.

REFERENCES


EXERCISES

E5.1-1 Derive an analytic expression for the input admittance \( Y_{IN} \) to a lossless, \( Z_0 \) transmission line that is \( \theta \) degrees long and terminated in an admittance \( Y_L \).

E5.3-1 Derive in terms of \( Y_0 \) and electrical length, \( \theta \), expressions for the susceptance of (a) open-circuited lossless transmission lines and (b) short-circuited lossless lines.

E5.3-2 A 50-\( \Omega \) transmission line has a matched generator with available average power of 100 W. What is the voltage 45\(^\circ\) from the load if the load is \((50 + j50) \Omega\)?

E5.4-1 A 50-\( \Omega \) microstrip transmission line has a load of \((50 - j50) \Omega\) that is to be matched using a shunt 50-\( \Omega \), open-circuit terminated stub. Since this method of tuning can be “printed” on the board and requires no “via” to connect it to ground, it is expected to be the most economical, reproducible, and vibration-resistant tuning method. Assume the load impedance is constant over the bandwidth.

a. Use the Smith chart to determine the location and electrical length of the stub needed for the matching.

b. Use the Smith chart to determine the VSWR at the edges of a \( \pm 10\% \) bandwidth.

E5.4-2 Can you find a 50-\( \Omega \) perfectly matched input solution to the problem in E5.4-1 using a properly chosen \( Z_0 \) and electrical length? If so, what are the values?

E5.4-3 A transistor has an input, which consists of 25 \( \Omega \) shunted by 6.4 pF. Use the Smith chart to design a matching network to 50 \( \Omega \) at 1 GHz using only a 50-\( \Omega \) transmission line and an open-circuited 50-\( \Omega \) shunt stub.

E5.4-4 Can you find a way to tune the amplifier input of E5.4-2 using only transmission lines that have a lower VSWR over the 900 to 1100 MHz bandwidth?

E5.6-1 What are the maximum and minimum resistances that will be seen along a 50-\( \Omega \) transmission line having a length of at least one-half wavelength and a load VSWR of 3?
E5.11-1 A patch antenna at 850 MHz is measured to have an impedance of 
\( 5 - j25 \) \( \Omega \). Use the Smith chart to design a matching network to 
50 \( \Omega \) as well as you can (using an approximate method is acceptable) 
using two in-line (cascaded) transmission line sections of appropriate 
characteristic impedances.

E5.11-2 Repeat Exercise 5.11-1, this time using lumped elements and the \( Q \) 
matching method.

E5.11-3 Repeat Exercise 5.11-2, this time using only a cascade section of a 
50-\( \Omega \) line and an open-circuited, shunt 50-\( \Omega \) line stub. Use the Smith 
chart to determine the length of the shunt stub and its location on the 
main line.

E5.11-4 Use a network simulator to compare the bandwidths for –20-dB 
return loss for the solutions that you obtained for Exercises 5.11-1, 
5.11-2, and 5.11-3.

E5.11-5 a. Can you find a way to tune the load in E5.4-1 using an approxi-
mate method that would satisfy the requirement for an all printed 
matching network and give a VSWR over the band no higher 
than that achieved in the solution of E5.4-1.

b. Explain why this matching method gives lower VSWR over the 
operating bandwidth.

E5.14-1 Only two reactive elements are needed to conjugately match any 
complex impedance load to any complex impedance generator. Demon-
strate that there is an unlimited number of matching networks 
possible by using 6 to 10 reactive elements (L’s and C’s) to transform 
10 to 50 \( \Omega \). Use a “random walk” on the Smith chart that starts at 
10 \( \Omega \) and eventually, within 6 to 10 elements, “arrives” at 50 \( \Omega \).

E5.14-2 a. Estimate the \( Q \) of a single \( LC \) matching network needed to trans-
form 7 to 50 \( \Omega \) at 1000 MHz.

b. Draw \( Q \) circles on the Smith chart and use them as a guide to 
design a matching network consisting of a series \( L \) and a shunt \( C \).

c. Change the reactances of the two tuning elements to what they 
become at 900 and 1100 MHz. Plot their respective contours on 
the Smith chart. They will not quite transform to 50 \( \Omega \) but to 
some other impedances. What are the VSWR values at these im-
pedances (at therefore at 900 and 1100 MHz)?

E5.14-3 Repeat E5.14-2, but this time use a three-section (six-element) tuner. 
Does this give a better match (lower VSWR) over the 900 to 1100 
MHz bandwidth?