Analysis of Low Pass Filter Using Nonhomogeneous Transmission Lines

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***Abstract*—Transmission lines are an important component in electrical engineering, which can be used to guide energy as well as information. Nonhomogeneous transmission lines, which have position varying quantities, can be used to design matching circuit, delay equalizer, filters VLSI interconnections, etc. In analysis of nonhomogeneous transmission lines, an approach based on method of moment is used. As a basis function, a constant function is used and as weighting function we used a delta function or collocation. In this work, we observed several cases such as lossless and lossy homogeneous transmission lines with matching and arbitrary load. These cases verified the algorithm developed in this work. The second example concerned with nonhomogeneous transmission lines, whose results conformed with those given in the literature. The last example consists of nonhomogeneous transmission lines in form of abruptly changing transmission lines. This structure is used to design a low pass filter. The calculated reflection and transmission factor show almost the same results as given with a commercial available software.**

Keywords*—filter, method of moment, nonhomogeneous transmission line, wave impedance

I. Introduction

Transmission lines are an important component in electrical engineering, which can be used to guide energy as well as information. At higher frequencies, radio frequency or microwave applications, they are designed as signal processing components, for examples as filters. In the theory of transmission lines several characteristic quantities (R', L', C' and G') are defined. Nonhomogeneous transmission lines, which have position varying quantities, can be used to design matching circuit [1], delay equalizer [2], filters [3], wave shaping [4], processing of analog signals [5] and VLSI interconnections [6].

In analysis of nonhomogeneous transmission lines, a governing differential equation with non constant coefficients is derived, which can not be solved in the same way like in analysis of homogeneous transmission lines. There are several approaches introduced, i.e. expansion of Taylor's series [7], expansion of Fourier's series [8] and application of method of moment [9]. In this work we use the same approach as described in [9]. Method of moment is an implementation for solving an integral equation, in which we have an unknown integrand. In the method of moment, a basis function is introduced to describe the unknown voltages or currents. By weighting or sampling the equation in several position, the integral equation can be converted into a system of linear equations. The solution of this matrix is the distribution of voltage and current along the transmission line. In this work, we observe several cases such as lossless and lossy homogeneous transmission lines with matching and arbitrary load. These cases should verify the algorithm developed in this work. The second example concerns with nonhomogeneous transmission lines. The last example consists of nonhomogeneous transmission lines in form of abruptly changing transmission lines. This structure is used to design a low pass filter. A computer code based on MATLAB is developed to calculate the reflection and transmission factor of such nonhomogeneous transmission lines.

II. Wave Equation of Nonhomogeneous Transmission Lines and its Solution

The theory of transmission lines gives the relationships between the voltage and current along the structure by the following equations

$$\frac{dV(z)}{dz} = -Z'(z)\,I(z) \qquad (1)$$

$$\frac{dI(z)}{dz} = -Y'(z)\,V(z) \qquad (2)$$

with position varying parameters $Z'(z) = R'(z) + j\omega L'(z)$ and $Y'(z) = G'(z) + j\omega C'(z)$.

Eqs. (1) and (2) lead to an nonhomogeneous differential equation of second order

$$\frac{d^2V(z)}{dz^2} - f(z)\frac{dV(z)}{dz} - \gamma^2(z)\,V(z) = 0 \qquad (3)$$

with $f(z) = (dZ'/dz)Z'$ and $\gamma^2 = Z'Y'$. The solution of Eq. (3) is not simple, and is available analytically only for some special functions of Z' and Y'. To give a solution for the problem, this paper takes the approach gived in [1], i.e. integrating Eqs. (1) and (2) leads to

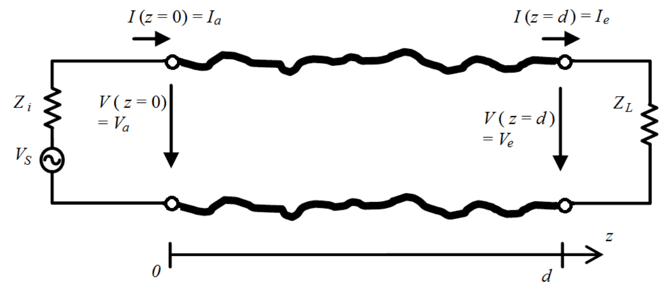

Fig. 1. Nonhomogeneous transmission line with source V_S and internal impedance Z_i and load Z_L.

$$V(z) = -\int_0^z Z'(z')\,I(z')dz' + C_1 \quad (4)$$

$$I(z) = -\int_0^z Y'(z')\,V(z')dz' + C_2 \quad (5)$$

The integration constants, C_1 and C_2, can be derived from the boundary conditions given in Fig.1, which are

$$C_1 = \frac{Z_L}{Z_i+Z_L}V_S + \frac{Z_i}{Z_i+Z_L}\int_0^d \left[Z'I - Z_L Y'V\right]dz'$$

$$C_2 = \frac{1}{Z_i+Z_L}V_S - \frac{1}{Z_i+Z_L}\int_0^d \left[Z'I - Z_L Y'V\right]dz'$$

Eqs. (4) and (5) are integrals of unknown current and voltage along the transmission line. It is worthy to express the unknown voltage and current using a combination of simply integrable functions with unknown amplitudes,

$$V(z) = \sum_{n=1}^{N} V_n f_n(z) \quad (6)$$

$$I(z) = \sum_{n=1}^{N} I_n g_n(z) \quad (7)$$

f_n and g_n are simple known functions, which are called basis functions, V_n and I_n are the unknown constants, and N is the number of approximating functions. In this case, we have $2 \times N$ unknowns.

By applying eqs. (6) and (7) into eqs. (4) and (5) we get a matrix equation

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}\begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{I} \end{pmatrix} = \begin{pmatrix} Z_L \\ 1 \end{pmatrix}\frac{V_S}{Z_i+Z_L} \quad (8)$$

with the unknown vectors $\boldsymbol{V}= [V_1\, V_2 \cdots V_N]^T$ and $\boldsymbol{I}= [I_1\, I_2 \cdots I_N]^T$, and known $1 \times N$ matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$, whose elements are

$$A_n = f_n + \frac{Z_i Z_L}{Z_i+Z_L}\int_0^d Y' f_n dz' \quad (9)$$

$$B_n = \int_0^z Z' g_n dz' - \frac{Z_i}{Z_i+Z_L}\int_0^d Z' g_n dz' \quad (10)$$

$$C_n = \int_0^z Y' f_n dz' - \frac{Z_L}{Z_i+Z_L}\int_0^d Y' f_n dz' \quad (11)$$

$$D_n = g_n + \frac{1}{Z_i+Z_L}\int_0^d Z' g_n dz' \quad (12)$$

In order to calculate the integrals in eqs. (9) to (12) numerically, the transmission line is divided, i.e. discretized, into N segments. In case of uniform discretization, the segment length Δz is equal to d/N. A significant simplification of the integration can be gained, if we set the basis function in each segment constant, as also described in [9], or

$$f_n(z) = g_n(z) = \begin{cases} 1 & \text{for } (n-1)\Delta z \le z \le n\Delta z \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

With the basis function in eq. (13), the integration along 0 to d or along 0 to z yields to zero except in a small observed segment n. Furthermore, if the transmission line is discretized fine enough, i.e. N is large, Δz becomes small enough, we can approximate the integration just by a simple multiplication between the segment length Δz and the mean value of Z' or Y', or just taking the value of Z' or Y' at the mid point of the segment z_n, so we get the values of A_n, B_n, C_n and D_n at the position z_m as follow

$$A_n(z_m) = \delta_{mn} + \frac{Z_i Z_L}{Z_i+Z_L}Y'(z_n)\Delta z \quad (14)$$

$$B_n(z_m) = Z'(z_n)U_{mn}\Delta z - \frac{Z_i}{Z_i+Z_L}Z'(z_n)\Delta z \quad (15)$$

$$C_n(z_m) = Y'(z_n)U_{mn}\Delta z - \frac{Z_L}{Z_i+Z_L}Y'(z_n)\Delta z \quad (16)$$

$$D_n(z_m) = \delta_{mn} + \frac{1}{Z_i+Z_L}Z'(z_n)\Delta z \quad (17)$$

z_m is the observation position, which can be chosen at the middle of the segment. To solve the problem uniquely, we can choose N observation positions, so that the number of equations is equal to the number unknowns.

δ_{mn} is the Kronecker function, and

$$U_{mn} = \begin{cases} 1 & \text{for } m > n \\ 1/2 & \text{for } m = n \\ 0 & \text{for } m < n \end{cases}$$

Which means, if the observation position on the right side of the integration boundary ($m > n$), we get the full integration. If the observation position is located exactly at the middle of the integration range, it yields a half of the result. And if the observation position at the left side of the integration range ($m < n$), the integration gives the value zero.

This procedure is a type of the method of moment [9], whose basis function uses pulse function and as test function is delta function used. This is also called collocation method.

III. Results

In this work, firstly we observe homogeneous transmission lines with matching and arbitrary loadings. As additional parameter we use lossless and lossy transmission lines. In the next example, we study the convergence of the segmentation in case of nonhomogeneous transmission lines. Furthermore we analyze a low pass filter as practical implementation of nonhomogeneous transmission lines.

A. Voltage distribution along homogeneous transmission lines

At first, a $0.6m$ length homogeneous transmission line is observed. The transmission line have a constant capacitance per unit length $C' = 66.7$ pF and inductance per unit length $L' = 0.167$ μH, so that a wave impedance of $Z_o = 50\,\Omega$ is obtained. The transmission line is connected with a load of $Z_L = 50\,\Omega$, and excited by a voltage source $V_S = 1\,V$ with a frequency of $f = 1$ GHz. In this example, we set $G' = 0$ and vary the value of resistance per unit length R'. Fig. 2 shows voltage distribution along the transmission line for different losses. For lossless case, we see a constant curve, which means there is no standing wave observed, the value of voltage standing wave ration (VSWR) is 1. We do not have any reflected waves. For lossy cases ($R' > 0$), the propagating waves experience attenuation along the transmission line. Fig. 2 illustrates, the larger the resistance per unit length, the smaller the amplitude of the voltage apart from the source.

Interestingly, in case of higher losses, for this 'matching' condition, we observe standing wave in form of small ripples. Such ripples must originate from superposition between incident and reflected waves. In this case, reflections indeed happen. With losses ($R' \neq 0$), the value of the wave impedance is no longer

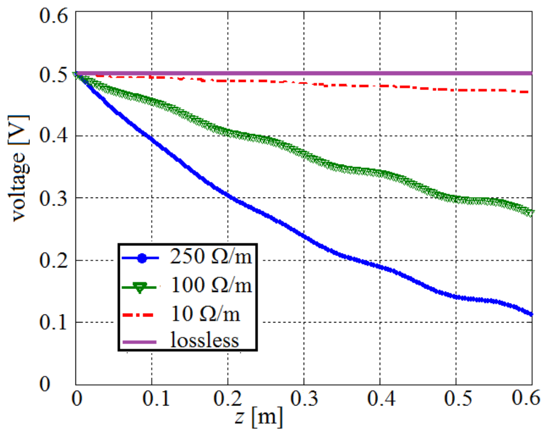

Fig. 2. Voltage distribution along transmission line with matching loading in dependence on resistance per unit length R'.

$50\,\Omega$. For example, with $R' = 250\,\Omega$/m, we get a complex wave impedance of $(50.3864 - j5.9196)\,\Omega$. This yields together with the load $Z_L = 50\,\Omega$ a reflection factor of $|r| = 0.059$, or a VSWR of 1.1254.
Now, the load is replaced with an impedance $Z_L = 20\,\Omega$. On a wave impedance of $50\,\Omega$, this leads to a reflection factor $r = -0.4286$ or VSWR$= 2.5$. For lossless case in Fig. 3, the standing wave pattern does not change along the transmission line, with a maximal voltage of $V_{max} = 0.7143$ V and a minimal voltage of $V_{min} = 0.2857$ V. At the load position, we have a minimum, because the load impedance is smaller than the wave impedance of the transmission line, as verified by the theory.

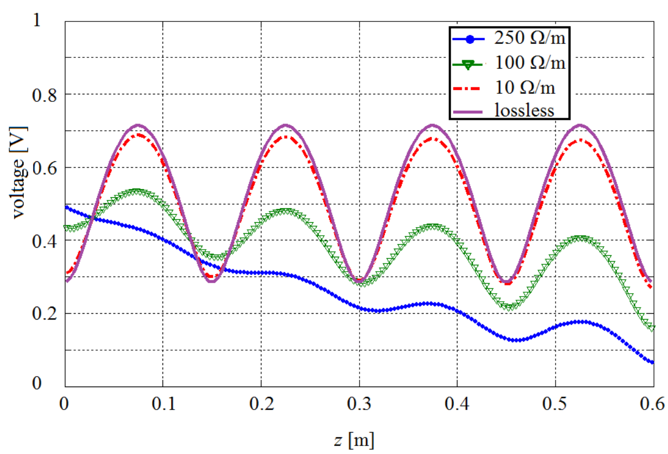

Fig. 3. Voltage distribution along transmission line with $Z_L = 20\Omega$ in dependence to resistance per unit length R'.

By considering losses in calculation, the standing wave pattern changes along the structure. We observe smaller VSWR at positions near to the source than near to the load. From the theory of lossy transmission lines, we learnt, the incident wave from the source to the load is attenuated, and after reflected by the load back to the source, the reflected wave is again attenuated. This makes the contribution of the reflected wave to the standing wave pattern near the source small as compared to the lossless case. So that for very lossy cases, near to the source the value of V_{max} is practically equal to the value of V_{min}.

B. Voltage distribution along nonhomogeneous transmission lines

In this section, for verification purposes the cases used in [9] are simulated by the codes developed here. The transmission line under observation is nonhomogeneous with a position-varying inductance per unit length, which increases linearly

$$L'(z) = L'_o\,(1 + k\,z/d)$$

Meanwhile, the capacitance per unit length decreases inversely with a similar rhythm as the inductance,

$$C'(z) = C'_o/\,(1 + k\,z/d)$$

with $C'_o = 66.7$ nF, $L'_o = 0.167$ μH and k =1. The length of the transmission line is d = 0.20 m. Other parameters are chosen for lossless conditions, $R'(z) = G'(z) = 0$.
With the data, the wave impedance becomes

$$Z_{o,non} = \sqrt{\frac{L'_o}{C'_o}}\left(1 + k\,\frac{z}{d}\right) = \left(1 + \frac{z}{d}\right)50\Omega$$

The simulation is performed with a voltage source $V_S = 1$ V, the frequency $f = 1$ GHz, and an internal impedance $Z_S = 50\,\Omega$. At the end of the nonhomogeneous transmission

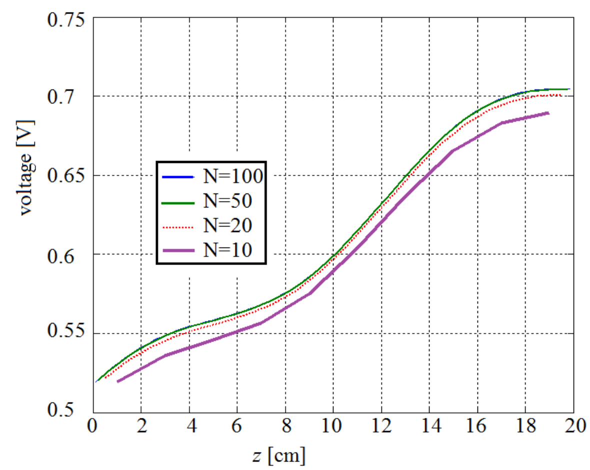

Fig. 4. Voltage distribution along nonhomogeneous transmission line for k = 1 and $Z_L = 100\Omega$ with different numbers of discretization N.

line, a load $Z_L = 100\,\Omega$ is connected, this is equal to the wave impedance $Z_{o,non}$ at the end position, $z = d$. So that we expect no reflection due to the load.
Fig. 4 gives the voltage distribution along the transmission line with different numbers of discretization. Theoretically, the higher the number of the segments, the more accurate the result obtained. However, the higher the number of the segments, the larger the computer memory (Random access memory/RAM) is needed and also the longer the calculation proceeds. By doubling the discretization from $N = 10$ to $N = 20$, we see the result changes relative significantly. This means, the result gained with $N = 10$ is not accurate. Moreover with $N = 50$ and $N = 100$ we are not able to distinguish the curve anymore,

which means, it converges.
The voltage distribution along the nonhomogeneous transmission lines is not constant even for matching load. However we see, the curve changes regularly.
For the second nonhomogeneous case, we use $k = 1.5$ and $Z_L = 150\,\Omega$. For this case, at the far end of the transmission line, the wave impedance has the value $Z_{o,non} = 125\,\Omega$, so that it is not a matching condition, so we expect a reflection will happen.
Fig. 5 verifies that a discretization of $N = 20$ reachs already the convergence. Moreover, we see in the voltage distribution a kind of oscillation with a minimal voltage at around $z = 10$ cm and a maximal voltage at around $z = 5$ cm.

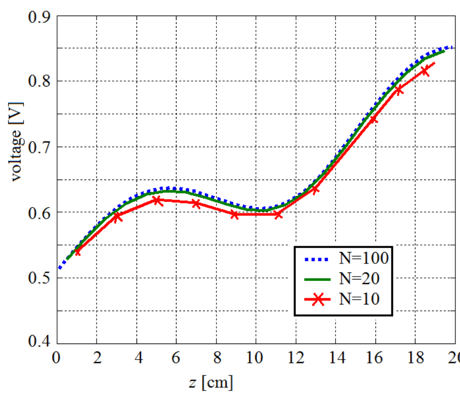

Fig. 5. Voltage distribution along nonhomogeneous transmission line for k = 1.5 and $Z_L = 150\,\Omega$.

C. Low pass filters based on nonhomogeneous transmission lines

In the last example we take the filter structure designed in [10]. The low pass filter constitutes of three transmission line pieces with different wave impedances and length (Fig. 6). Here, we use the same impedance values as in [10], but the lengths of the transmission lines are three times longer than in [10], because we used there a microstrip structure with relative permittivity $\varepsilon_r = 9$.

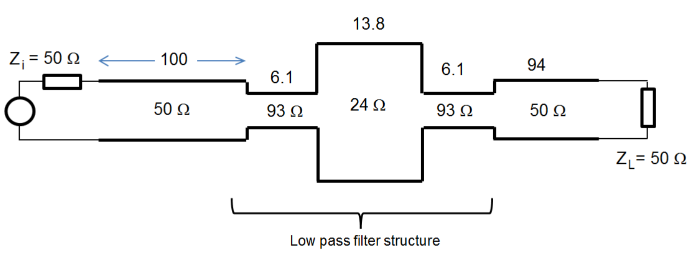

Fig. 6. Low pass filter with two 50 Ω feeding lines, all length dimensions in mm.

On the source and load sides, two connecting transmission lines with a wave impedance of 50 Ω are connected, and we connect an internal impedance $Z_S = 50\,\Omega$ and a load impedance $Z_L = 50\,\Omega$, so that both sides are in matching condition.
Our target here is to analyze the low pass filter structure in the frequency range between 1 GHz and 8 GHz. We calculate the reflection factor (S_{11}) and transmission factor (S_{21}) of the filter. The reflection emerges not due to the load, but rather due to the nonhomogeneous structure of the transmission line used (here abrupt changes of the impedances). This can be verified later, that we have a constant voltage distribution along the connecting line on the load side. It is indeed, because we have there just a wave propagating to the load. However along the connecting line on the source side, we expected a standing wave pattern, having a maximum and minimum voltage. From this pattern we can calculate the VSWR, and then the reflection factor.

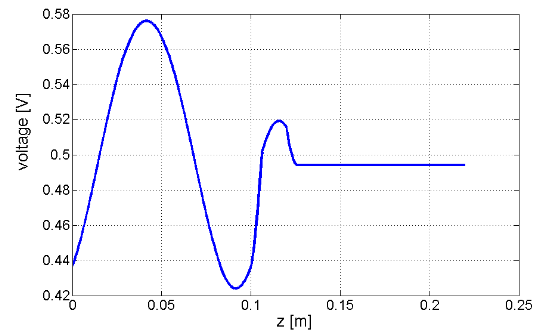

Fig. 7. Voltage distribution along the low pass filter structure for at frequency 1.5 GHz.

Fig. 7 shows the voltage distribution along the transmission line at the frequency 1.5 GHz. The curve on the load side is constant, this is the voltage wave towards the load with the value $V_t = 0.4942$ V. On the source side, we have a standing wave pattern with $V_{max} = 0.576$ V and $V_{min} = 0.424$ V, which yields a VSWR of 1.3586 or a reflection factor of 0.1521 (-16.36 dB). On the source side, we can calculate the voltage wave propagating to the input side of the filter with $V_{inc} = 0.5(V_{max} + V_{min}) = 0.5$ V, so that the transmission factor can be calculated to $t = 0.4942/0.5 = 0.9883$ (-0.1022 dB).

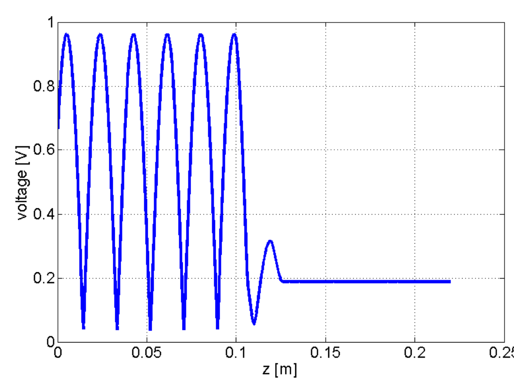

Fig. 8. Voltage distribution along the low pass filter structure for at frequency 8 GHz.

Whereas Fig. 8 gives the voltage distribution for the frequency 8 GHz. At this frequency, the standing wave pattern is more obvious. The maximal voltage of 0.9621 V and the minimal voltage of 0.0369 V deliver an VSWR of 26.0749 or a reflection factor 0.9261 (-0.6668 dB). With the voltage at the load side $V_t = 0.1884$ V and the incident voltage on the source

again $V_{inc} = 0.5(V_{max} + V_{min}) = 0.5$ V, so that the transmission factor can be calculated to $t = 0.1884/0.5 = 0.3767$ (-8.4801 dB).
We can also check that the condition $r^2 + t^2 = 1$ is fulfilled, which means, that all incident power is converted into reflected and transmitted power, because we neglected any losses in this calculation.

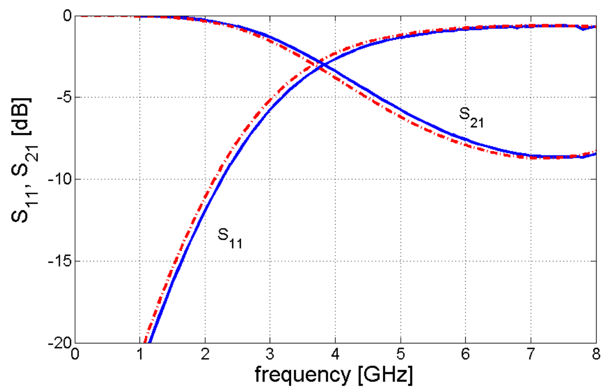

Fig. 9. Scattering parameters, solid lines: this work, dashed lines: Sonnet v.13.

By varying the frequency from 1 GHz to 8 GHz, we can calculate the reflection and transmission factor over this frequency range. The result is depicted in Fig. 9. As comparison, we take the low pass filter designed in [10] which is calculated with Sonnet [11]. We see a very good similarity between both result.

IV. Conclusion

The integral equation method implemented by means of method of moment yields very good results. Several canonical problems, such as homogeneous transmission lines with and without losses, verify this computer simulations. The numerical results coincide with that yielded by theoretical approach. Observation of smooth and abruptly changed transmission lines reveals also this powerful numerical calculation. At last, abruptly changing nonhomogeneous transmission line, which presents itself as low pass filter, was considered. Comparing the results with those obtained by a commercial software shows very good coincidences.

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